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A unified class of analytic functions involving a generalization of the Srivastava–Attiya operator

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ABSTRACT

In this paper, we present a unified class of analytic functions defined by a new convolution operator $J^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}$ introduced recently by Srivastava and Gaboury (2014) which generalizes the well-known Srivastava–Attiya operator investigated by Srivastava and Attiya (2007). We derive coefficient inequalities, growth and distortion theorems, extreme points and Fekete–Szegö problem for this new function class.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{C}$ the class of convex functions $f(z) \in \mathcal{A}$ that are convex in $\mathbb{U}$.

The Srivastava–Attiya operator is defined as [22] (see also [3,16,31]):

$$J_{s,a}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s a_k z^k \tag{1.2}$$

where $z \in \mathbb{U}, a \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ and $f \in \mathcal{A}$.

In fact, the linear operator $J_{s,a}(f)$ can be written as

$$J_{s,a}(f)(z) := G_{s,a}(z) * f(z) \tag{1.3}$$

in terms of Hadamard product (or convolution) where $G_{s,b}(z)$ is given by

$$G_{s,a}(z) := (1+a)^s [\Phi(z,s,a) - a^{-s}] \quad (z \in \mathbb{U}). \tag{1.4}$$

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The function $\Phi(z,s,a)$ involved in the right-hand side of (1.4) is the well-known Hurwitz–Lerch zeta function defined by (see, for example, [23, p. 121 *et seq.*]; see also [18] and [24, p. 194 *et seq.*])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.5}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \quad \text{when } |z| = 1).$$

Recently, a new family of λ-generalized Hurwitz–Lerch zeta function was investigated by Srivastava [21] (see also [19,20,26,28,32]). Srivastava considered the following function:

$$\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda) = \frac{1}{\lambda\,\Gamma(s)} \cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\, H^{2,0}_{0,2}\left[(a+n)b^{\frac{1}{\lambda}} \Big| \overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right] \frac{z^n}{n!} \quad (\min\{\Re(a),\Re(s)\} > 0; \; \Re(b) > 0;\; \lambda > 0), \tag{1.6}$$

where

$$\left(\lambda_j \in \mathbb{C} \;\; (j=1,\ldots,p) \text{ and } \mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^- \;(j=1,\ldots,q);\; \rho_j > 0 \;(j=1,\ldots,p); \sigma_j > 0 \;(j=1,\ldots,q);\; 1+\sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j \geqq 0\right)$$

and the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p}\rho_j^{-\rho_j}\right)\cdot\left(\prod_{j=1}^{q}\sigma_j^{\sigma_j}\right).$$

Here, and for the remainder of this paper, $(\lambda)_\kappa$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa = n \in \mathbb{N};\; \lambda \in \mathbb{C}) \\ 1 & (\kappa = 0;\; \lambda \in \mathbb{C}\setminus\{0\}), \end{cases} \tag{1.7}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [29, p. 21 *et seq.*]).

Definition 1.1. The H-function involved in the right-hand side of (1.6) is the well-known Fox's H-function [12, Definition 1.1] (see also [27,29]) defined by

$$H^{m,n}_{\mathfrak{p},\mathfrak{q}}(z) = H^{m,n}_{\mathfrak{p},\mathfrak{q}}\left[z \Big| \begin{matrix} (a_1,A_1),\ldots,(a_\mathfrak{p},A_\mathfrak{p}) \\ (b_1,B_1),\ldots,(b_\mathfrak{q},B_\mathfrak{q}) \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \Xi(s) z^{-s}\,ds \quad (z \in \mathbb{C}\setminus\{0\};\; |\arg(z)| < \pi), \tag{1.8}$$

where

$$\Xi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\cdot\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=n+1}^{\mathfrak{p}}\Gamma(a_j+A_js)\cdot\prod_{j=m+1}^{\mathfrak{q}}\Gamma(1-b_j-B_js)},$$

an empty product is interpreted as $1, m, n, \mathfrak{p}$ and $\mathfrak{q}$ are integers such that $1 \leqq m \leqq \mathfrak{q}, 0 \leqq n \leqq \mathfrak{p}, A_j > 0 \;(j=1,\ldots,\mathfrak{p}), B_j > 0 \;(j=1,\ldots,\mathfrak{q}), a_j \in \mathbb{C} \;(j=1,\ldots,\mathfrak{p}), b_j \in \mathbb{C} \;(j=1,\ldots,\mathfrak{q})$ and $\mathcal{L}$ is a suitable Mellin–Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(b_j+B_js)\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\{\Gamma(1-a_j+A_js)\}_{j=1}^{n}.$$

It is worthy to mention that using the fact that [21, p. 1496, Remark 7]

$$\lim_{b\to 0}\left\{H^{2,0}_{0,2}\left[(a+n)b^{\frac{1}{\lambda}}\Big|\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\right\} = \lambda\,\Gamma(s) \qquad (\lambda > 0), \tag{1.9}$$

Eq. (1.6) reduces to

$$\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;0,\lambda) := \Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s\cdot\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\frac{z^n}{n!}. \tag{1.10}$$