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Convergence of relaxation iterative methods for saddle point problem ☆

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ABSTRACT

In this paper, we first provide convergence results of three relaxation iterative methods for solving saddle point problem. Next, we propose how to find near optimal parameters for which preconditioned Krylov subspace method performs nearly best when the relaxation iterative methods are applied to the preconditioners of Krylov subspace method. Lastly, we provide efficient implementation for the relaxation iterative methods and efficient computation for the preconditioner solvers. Numerical experiments show that the MIAOR method and the BiCGSTAB with MAOR preconditioner using near optimal parameters perform more than twice faster than the GSOR method.

1. Introduction

In this paper, we consider relaxation iterative methods for solving the following saddle point problem

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, and $B \in \mathbb{R}^{m\times n}$ is a matrix of full column rank with $m \geqslant n$. This problem (1) appears in many different scientific applications, such as constrained optimization [13], the finite element approximation for solving the Navier–Stokes equation [7,8], and the constrained least squares problems and generalized least squares problems [1,3,16,18]. There have been many methods for solving the saddle point problem (1). Golub et al. [9] proposed the Successive OverRelaxation-like (SOR-like) method, Bai et al. [3] proposed the Generalized Successive OverRelaxation (GSOR) method and the Generalized Inexact Accelerated OverRelaxation (GIAOR) method, Bai and Wang [4] proposed the Parameterized Inexact Uzawa (PIU) method, Li et al. [10] proposed the Accelerated OverRelaxation-like (AOR-like) method, Shao et al. [11] proposed the modified SOR-like method, Zheng et al. [20] and Darvishi and Hessari [6] proposed and studied the Symmetric Successive OverRelaxation-like (SSOR-like) method, Wu et al. [14] proposed the modified SSOR-like method, Zhang and Lu [19] and Chao et al. [5] studied the Generalized Symmetric Successive OverRelaxation (GSSOR) method, Yun [17] proposed the Uzawa Symmetric Accelerated OverRelaxation (Uzawa-SAOR) method, and so on.

For simplicity of exposition, some notation and definitions are presented. For a vector x, x^* denotes the complex conjugate transpose of the vector x. $\lambda_{min}(H)$ and $\lambda_{max}(H)$ denote the minimum and maximum eigenvalues of the Hermitian matrix H, respectively. For a square matrix G, $N(G)$ denotes the null space of G, $\sigma(G)$ denotes the set of all eigenvalues of G, and $\rho(G)$ denotes the spectral radius of G.

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The purpose of this paper is to provide convergence results of three relaxation iterative methods and an application of the relaxation iterative methods to the preconditioners of Krylov subspace method for solving the saddle point problem (1). This paper is organized as follows. In Section 2, we provide convergence result for the Modified Inexact Accelerated OverRelaxation-like (MIAOR) method. In Section 3, we provide an improved convergence result for the GSSOR method. In Section 4, we provide convergence result for the UnSymmetric Successive OverRelaxation-like (USSOR) method. In Section 5, we propose how to find near optimal parameters for which preconditioned Krylov subspace method performs nearly best when the relaxation iterative methods are applied to the preconditioners of Krylov subspace method. In Section 6, we provide efficient implementation for three relaxation iterative methods, and then we provide efficient computation for the preconditioner solvers which are one of the main computational kernels of Krylov subspace method. In Section 7, numerical experiments are carried out to examine the effectiveness of three relaxation iterative methods studied in this paper. Lastly, some conclusions are drawn.

2. Convergence result for the MIAOR method

For the coefficient matrix of the saddle point problem (1), we consider the following splitting

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = D - L - U, \tag{2}$$

where

$$D = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ B^T & \alpha Q \end{pmatrix}, \quad U = \begin{pmatrix} P - A & -B \\ 0 & \beta Q \end{pmatrix},$$

where $P \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix which approximates A, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix which approximates the approximated Schur complement matrix $B^TP^{-1}B$, and $\alpha + \beta = 1$ with $0 \leqslant \alpha \leqslant 1$. Then the modified inexact AOR-like method (MIAOR) for solving the saddle point problem (1) is defined by

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = H(\alpha, r, \omega)\begin{pmatrix} x_k \\ y_k \end{pmatrix} + M(\alpha, r, \omega)\begin{pmatrix} f \\ -g \end{pmatrix}, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where

$$H(\alpha, r, \omega) = (D - rL)^{-1}((1-\omega)D + (\omega - r)L + \omega U) = \begin{pmatrix} P & 0 \\ -rB^T & (1-\alpha r)Q \end{pmatrix}^{-1}\begin{pmatrix} P - \omega A & -\omega B \\ (\omega - r)B^T & (1-\alpha r)Q \end{pmatrix},$$

$$M(\alpha, r, \omega) = \omega(D - rL)^{-1} = \omega\begin{pmatrix} P & 0 \\ -rB^T & (1-\alpha r)Q \end{pmatrix}^{-1},$$

$\omega > 0$ and $(1-\alpha r) \neq 0$. Notice that $(1-\alpha r) \neq 0$ is required for the MIAOR method to be well defined. If $P = A$, then the MIAOR method reduces to the modified AOR-like method (MAOR). If $P = A$ and $\omega = r$, then the MIAOR method reduces to the modified SOR-like method (MSOR).

Let λ be an eigenvalue of $H(\alpha, r, \omega)$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ be the corresponding eigenvector. Then we have

$$\begin{pmatrix} P - \omega A & -\omega B \\ (\omega - r)B^T & (1-\alpha r)Q \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} P & 0 \\ -rB^T & (1-\alpha r)Q \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

or equivalently,

$$\begin{aligned} &(1-\lambda)Pu - \omega Au = \omega Bv, \\ &(\omega - r + \lambda r)B^Tu = (\lambda - 1)(1-\alpha r)Qv. \end{aligned} \tag{4}$$

Since A is a symmetric positive definite matrix and B is a matrix of full column rank, it can be easily shown that $\lambda \neq 1$ and $u \neq 0$. We first introduce the following lemma in [4] with a simple proof.

Lemma 2.1 [4]. *Assume that the matrices A, P, Q and B are defined as above. Define $\eta(u) = \frac{u^*Au}{u^*Pu}$ and $\mu(u) = \frac{u^*BQ^{-1}B^Tu}{u^*Pu}$ for a nonzero vector u. Then*

(a) $0 < \lambda_{min}(P^{-1}A) \leqslant \eta(u) \leqslant \lambda_{max}(P^{-1}A) = \rho(P^{-1}A)$,
(b) $\mu_{min} = \min_{u\notin N(B^T)}\mu(u) = \min\{\lambda | \lambda \in \sigma(Q^{-1}B^TP^{-1}B)\}$,
(c) $\mu_{max} = \max_{u\notin N(B^T)}\mu(u) = \max\{\lambda | \lambda \in \sigma(Q^{-1}B^TP^{-1}B)\} = \rho(Q^{-1}B^TP^{-1}B)$.

Proof. Since $P^{-1}A$ is similar to $P^{-\frac{1}{2}}AP^{-\frac{1}{2}}$ which is symmetric positive definite, it is easy to show that

$$\lambda_{min}(P^{-1}A) = \min_{u\neq 0}\eta(u) \text{ and } \lambda_{max}(P^{-1}A) = \max_{u\neq 0}\eta(u).$$

From these relations, part (a) follows. Since $P^{-1}BQ^{-1}B^T$ is similar to $P^{-\frac{1}{2}}BQ^{-1}B^TP^{-\frac{1}{2}}$ which is symmetric positive semidefinite,