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Oscillation results for difference equations with several oscillating coefficients

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ABSTRACT

This paper presents a new sufficient condition for the oscillation of all solutions of difference equations with several deviating arguments and oscillating coefficients. Corresponding difference equations of both retarded and advanced type are studied. Examples illustrating the results are also given.

1. Introduction

In the present paper, we study the oscillatory behavior of the solutions of the retarded difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \tag{E_R}$$

where $\mathbb{N} \ni m \geqslant 2,\ p_i,\ 1 \leqslant i \leqslant m$, are sequences of real numbers and $\{\tau_i(n)\}_{n\in\mathbb{N}_0}, 1 \leqslant i \leqslant m$, are sequences of integers such that

$$\tau_i(n) \leqslant n-1, \quad n \in \mathbb{N}_0, \quad \text{and } \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \leqslant i \leqslant m \tag{1.1}$$

and the (dual) advanced difference equation

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, \quad n \in \mathbb{N}, \tag{E_A}$$

where $\mathbb{N} \ni m \geqslant 2,\ p_i,\ 1 \leqslant i \leqslant m$, are sequences of real numbers and $\{\sigma_i(n)\}_{n\in\mathbb{N}},\ 1 \leqslant i \leqslant m$, are sequences of integers such that

$$\sigma_i(n) \geqslant n+1, \quad n \in \mathbb{N}, \quad 1 \leqslant i \leqslant m. \tag{1.2}$$

Here, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Also, as usual, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

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By a *solution* of (E_R), we mean a sequence of real numbers $\{x(n)\}_{n\geqslant -w}$ which satisfies (E_R) for all $n\in\mathbb{N}_0$. Here,

$$w:=-\min_{\substack{n\in\mathbb{N}_0\\ 1\leqslant i\leqslant m}}\tau_i(n)\in\mathbb{N}_0.$$

It is clear that, for each choice of real numbers $c_{-w},c_{-w+1},\ldots,c_{-1},c_0$, there exists a unique solution $\{x(n)\}_{n\geqslant -w}$ of (E_R) which satisfies the initial conditions $x(-w)=c_{-w},x(-w+1)=c_{-w+1},\ldots,x(-1)=c_{-1},x(0)=c_0$.

By a *solution* of the advanced difference equation (E_A), we mean a sequence of real numbers $\{x(n)\}_{n\in\mathbb{N}_0}$ which satisfies (E_A) for all $n\in\mathbb{N}$.

A solution $\{x(n)\}_{n\geqslant -w}\left[\{x(n)\}_{n\in\mathbb{N}_0}\right]$ of (E_R) $[(E_A)]$ is called *oscillatory* (around zero), if for any positive integer $n_0\geqslant -w$ $[n_0\geqslant 0]$ there exist $n_1,n_2\geqslant n_0$ such that $x(n_1)x(n_2)\leqslant 0$. Otherwise, the solution is said to be *nonoscillatory*.

In the last few decades, the oscillatory behavior of all solutions of difference equations has been extensively studied when the coefficients $p_i(n)$ are nonnegative. However, for the general case when $p_i(n)$ are allowed to oscillate, it is difficult to study the oscillation of (E_R) $[(E_A)]$, since the difference $\Delta x(n)[\nabla x(n)]$ of any nonoscillatory solution of (E_R) $[(E_A)]$ is in general oscillatory. Therefore, the results on oscillation of difference and differential equations with oscillating coefficients are relatively scarce. Thus, a small number of paper are dealing with this case. See, for example, [1–16] and the references cited therein.

In 1992, Qian et al. [12], in 2000, Yu and Tang [16] and in 2001, Tang and Cheng [13] derived oscillation conditions for a special case of equation (E_R) the following equation with one oscillating coefficient and constant delay of the form

$$x_{n+1}-x_n+p_nx_{n-k}=0,\quad n\in\mathbb{N}_0,$$

while in 1996 Yan and Yan [15] studied the difference equation with several oscillating coefficients of the form

$$x_{n+1}-x_n+\sum_{i=1}^{m}p_i(n)x_{n-k_i(n)}=0,\quad n\in\mathbb{N}_0,$$

under some additional conditions on the oscillating coefficients.

For equations (E_R) and (E_A) with several oscillating coefficients, very recently, Bohner et al. [2,3] and Berezansky et al. [1] established the following theorems.

Theorem 1.1 (See [2, Theorem 2.4]). *Assume (1.1) and that the sequences τ_i* are increasing for all $i\in\{1,\ldots,m\}$. *Suppose also that for each $i\in\{1,\ldots,m\}$ there exists a sequence $\{n_i(j)\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}n_i(j)=\infty$ and*

$$p_k(n)\geqslant 0 \text{ for all } n\in\bigcap_{i=1}^{m}\left\{\bigcup_{j\in\mathbb{N}}[\tau(\tau(n_i(j))),n_i(j)]\cap\mathbb{N}\right\}\neq\emptyset,\quad 1\leqslant k\leqslant m,\tag{1.3}$$

where

$$\tau(n)=\max_{1\leqslant i\leqslant m}\tau_i(n),\quad n\in\mathbb{N}_0.\tag{1.4}$$

If, moreover

$$\limsup_{j\to\infty}\sum_{i=1}^{m}\sum_{q=\tau(n(j))}^{n(j)}p_i(q)>1,\tag{1.5}$$

where $n(j)=\min\{n_i(j):1\leqslant i\leqslant m\}$, then all solutions of (E_R) oscillate.

Theorem 1.2 (See [2, Theorem 3.4]). *Assume (1.2) and that the sequences σ_i are increasing for all $i\in\{1,\ldots,m\}$*. Suppose also that for each $i\in\{1,\ldots,m\}$ *there exists a sequence $\{n_i(j)\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}n_i(j)=\infty$ and*

$$p_k(n)\geqslant 0\quad\text{for all } n\in\bigcap_{i=1}^{m}\left\{\bigcup_{j\in\mathbb{N}}[n_i(j),\sigma(\sigma(n_i(j)))]\cap\mathbb{N}\right\}\neq\emptyset,\quad 1\leqslant k\leqslant m,\tag{1.6}$$

where

$$\sigma(n)=\min_{1\leqslant i\leqslant m}\sigma_i(n),\quad n\in\mathbb{N}.\tag{1.7}$$

If, moreover

$$\limsup_{j\to\infty}\sum_{i=1}^{m}\sum_{q=n(j)}^{\sigma(n(j))}p_i(q)>1,\tag{1.8}$$

where $n(j)=\max\{n_i(j):1\leqslant i\leqslant m\}$, then all solutions of (E_A) oscillate.

Theorem 1.3 (See [3, Theorem 2.1]). *Assume (1.1) and that the sequences τ_i are increasing for all $i\in\{1,\ldots,m\}$. Suppose also that for each $i\in\{1,\ldots,m\}$ there exists a sequence $\{n_i(j)\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}n_i(j)=\infty$,*