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The convergence of wavelet expansion with divergence-free properties in vector-valued Besov spaces ☆

Junjian Zhao

Department of Mathematics, Tianjin Polytechnic University, No. 399 Binshuixi Street, Xiqing District, Tianjin 300387, PR China

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ABSTRACT

Using the biorthogonal B-spline wavelets, we investigate the convergence property of wavelets expansion in the related vector-valued Besov spaces. Especially, divergence-free and non divergence-free wavelets are added and discussed. As a by-product, characterization of relevant Besov spaces is given as well. It is noted that the key point for characterization is to prove the convergence of projection operators in relevant Besov spaces. Besides, we can get convergence and characterization at the same time.

1. Introduction

Wavelets are widely used in signal and image processing as well as numerical simulation. Divergence-free wavelets [13] are especially used to represent the incompressible flows. Besov spaces contain a large number of fundamental spaces, such as Sobolev spaces, Hölder spaces, Lipschitz spaces and etc [6,7]. And they are frequently used in certain PDEs as the solution spaces. It is known that characterizing functional spaces, especially by using wavelet bases, is important in both theory and applications. Bittner and Urban [1] study the following standard Besov spaces and the related vector Besov spaces.

Let $0 < p,\ q \leqslant \infty,\ s > 0$ and $\lfloor s \rfloor$ stands for the largest integer less than or equals to s,

$$B^s_{p,q}(\mathbb{R}^n) := \{f \in L_p(\mathbb{R}^n) : |f|_{B^s_{p,q}(\mathbb{R}^n)} < \infty\}.$$

Here, $|f|_{B^s_{p,q}(\mathbb{R}^n)} := \|(2^{js}\omega^M_p(f, 2^{-j}))_{j\in\mathbb{Z}}\|_{\ell_q}$ with $M \geqslant \lfloor s \rfloor + 1$ and $\omega^M_p(f, 2^{-j})$ denotes the Mth order smooth modulus of a function f, defined by $\sup_{|h|\leqslant 2^{-j}}\|\Delta^M_h f(\cdot)\|_{L_p(\mathbb{R}^n)}$ as usual. The classical difference operator Δ_h is defined by $\Delta_h f(\cdot) := f(\cdot + h) - f(\cdot)$, as well as $\Delta^M_h f = \Delta_h(\Delta^{M-1}_h f)$ for a positive integer $M > 1$. The Besov (quasi-) norm is given by $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|f\|_{L_p(\mathbb{R}^n)} + |f|_{B^s_{p,q}(\mathbb{R}^n)}$ and two integers $M, M' > s$ yield equivalent norms [2, Remark 3.2.2]. Based on quadratic or cubic Hermite multiwavelets, the authors of [1] characterize $B^s_{p,q}$ by using sequence norms

$$\|a\|_{\ell_p} := \|(a_{j_0,k})_{k\in\mathbb{Z}^n}\|_{\ell_p}, \quad \|b\|_{\ell^s_{p,q}} := \|(2^{j(s+\frac{n}{2}-\frac{n}{p})}\|(b_{j,k})_{k\in\mathbb{Z}^n}\|_{\ell_p})_{j\geqslant j_0}\|_{\ell_q}$$

for $a = (a_k)_{k\in\mathbb{Z}^n} \in \ell_p,\ b = (b_{j,k})_{j\geqslant j_0,\ k\in\mathbb{Z}^n} \in \ell^s_{p,q}$.

However, due to the regularity restrictions of the Hermite splines, their characterization requires $\frac{1}{p} < s < min\{3, 1 + \frac{1}{p}\}$ in quadratic case and $1 + \frac{1}{p} < s < min\left\{4, 2 + \frac{1}{p}\right\}$ in cubic one (e.g. [1,12,14]). In [11], we remove that restriction of s by using the

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E-mail addresses: zhaojunjian@emails.bjut.edu.cn, zhaojunjian@tjpu.edu.cn

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B-spline wavelets with weak duals introduced in [9]. Note that [11] only discussed 1-dimensional cases, without discussing vector cases or even divergence-free ones, and [1] only discussed interpolatory cases without the study of the convergence of vector-valued wavelets expansion. Hence, we will accomplish it by using the biorthogonal B-spline wavelets in this paper. So, one of the main result of this paper is to prove the convergence of divergence-free and non divergence-free wavelets expansion in vector fields.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the set of positive integers, the set of integers and the set of real numbers, respectively. $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$. Through out this paper, we use $A \lesssim B$ to abbreviate that A is bounded by a constant multiple of B, $A \gtrsim B$ is defined as $B \lesssim A$ and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. Write

$$\langle f, g\rangle = \int_\Omega f(t)\overline{g(t)}dt$$

for $f \in L_p(\Omega)$, $g \in L_{p'}(\Omega)$ with Lebesgue measurable set $\Omega \subseteq \mathbb{R}^n$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 \leqslant p \leqslant \infty$. For a Lebesgue measurable function f, the support of f means the set $\mathrm{Supp}(f) := \{x \in \mathbb{R} : f(x) \neq 0\}$, which is well-defined up to a set of measure 0.

Define $f_{n,k}(\cdot) := 2^{\frac{n}{2}} f(2^n \cdot -k)$ through out this paper except for special explanation. The classical Fourier transform is given by $\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx$ for $f \in L_1$ and a standard extension in other cases. A Multiresolution Analysis (MRA) is defined below, which is a sequence of approximated spaces allowing the construction of wavelets.

Definition 1.1 (*MRA*). A Multiresolution Analysis of $L_2(\mathbb{R})$ is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ verifying:

(i) $\forall j,\ V_j \subseteq V_{j+1};\ \bigcap_j V_j = \{0\}$ and $\overline{\bigcup_j V_j} = L_2(\mathbb{R})$;
(ii) $f \in V_j \Longleftrightarrow f(2\cdot) \in V_{j+1}$;
(iii) There exists a function $\phi \in V_0$ such that the family $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ form a (Riesz) basis of V_0.

We consider the classical r-order B-spline scaling function as:

$$\phi_r(\cdot) := \underbrace{\chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]}}_{r\ \ functions}(\cdot), \tag{1.1}$$

i.e., r convolutions of the characteristic function on the interval [0,1] with

$$\hat{\phi}_r(2\xi) = m_0(\xi)\hat{\phi}_r(\xi) = e^{-\frac{ik\xi}{2}}\left(cos\frac{\xi}{2}\right)^r\hat{\phi}_r(\xi)$$

for $k = 0$ if r is even and $k = 1$ if r is odd. The dual function $\tilde{\phi}_{r,\tilde{r}}$ of ϕ_r are defined by

$$\widehat{\tilde{\phi}_{r,\tilde{r}}(x)}(\xi) := (2\pi)^{-\frac{1}{2}}\prod_{j=1}^{\infty}\tilde{m}_0(2^{-j}\xi) \tag{1.2}$$

and ${}_{r,\tilde{r}}\tilde{m}_0(\xi) := e^{-\frac{ik\xi}{2}}(cos\frac{\xi}{2})^{\tilde{r}}\sum_{l=0}^{N}\begin{pmatrix} N-1+l \\ l\end{pmatrix}(sin\frac{\xi}{2})^{2l}$ for $\tilde{r} \geqslant 1$, $r+\tilde{r} = 2N$ is even, $k = 0$ if r is even and $k = 1$ if r is odd. And the wavelets with their duals are given by

$$\hat{\psi}_{r,\tilde{r}}(\xi) := e^{i\frac{\xi}{2}}\overline{{}_{r,\tilde{r}}\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)}\hat{\phi}_r\left(\frac{\xi}{2}\right), \quad \hat{\tilde{\psi}}_{r,\tilde{r}}(\xi) := e^{i\frac{\xi}{2}}\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\widehat{\tilde{\phi}_{r,\tilde{r}}}\left(\frac{\xi}{2}\right),$$

respectively. The above is the classical biorthogonal B-spline wavelets [3].

We are interested in here is to discuss divergence-free wavelets expansion in Besov spaces, so we start with the theory of divergence-free wavelets (specific forms will be shown in Section 4 and see [13] for more details). The construction of compactly supported biorthogonal divergence-free wavelets in $(L^2(\mathbb{R}^n))^n$ was originally derived by Lemarié–Rieusset [10], it is based on two different MRAs of $L^2(\mathbb{R})$ related by differentiation and integration.

Theorem 1.1. *Let $\{V_j^1\}$, a 1-dimensional MRA with a derivable scaling function φ_1 and a wavelet ψ_1, be given. Then, we can build a MRA $\{V_j^0\}$ with a scaling function φ_0 and a wavelet ψ_0 verifying*

$$V_0^1 = \overline{span}\{\varphi_1(x-k), k \in \mathbb{Z}\}, \quad V_0^0 = \overline{span}\{\varphi_0(x-k), k \in \mathbb{Z}\}$$

and

$$\varphi_1'(\cdot) = \varphi_0(\cdot) - \varphi_0(\cdot - 1), \quad \psi_1'(\cdot) = 4\psi_0(\cdot).$$

The relation of refinement polynomials is $m_0(\xi) = \frac{2}{1+e^{-i\xi}}m_1(\xi)$. The corresponding duals are $\tilde{\varphi}_1,\ \tilde{\varphi}_0,\ \tilde{\psi}_1,\ \tilde{\psi}_0$ satisfying

$$\tilde{\varphi}_0'(\cdot) = \tilde{\varphi}_1(\cdot+1) - \tilde{\varphi}_1(\cdot), \quad \tilde{\psi}_0'(\cdot) = -4\tilde{\psi}_1(\cdot)$$

with the refinement polynomials is $\tilde{m}_0(\xi) = \frac{1+e^{i\xi}}{2}\tilde{m}_1(\xi)$.

Luckily, the B-spline functions defined in (1.1) and (1.2) satisfy Theorem 1.1.