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Center problem in the center manifold for quadratic and cubic differential systems in $\mathbb{R}^3$ ☆

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ABSTRACT

We obtain necessary and sufficient conditions for the existence of a center on a local center manifold for three six-parameter families of quadratic systems on $\mathbb{R}^3$. We also give a positive answer to the conjecture posed in Mahdi (2013) for a special class of systems, called the Moon–Rand systems in the particular case when $\lambda = 2$ and f is a homogeneous cubic polynomial. We also solve the center problem for a natural generalization of the above mentioned Moon–Rand systems.

1. Introduction and statement of the main results

Assume that we have a polynomial system of differential equations in $\mathbb{R}^3, \dot{u} = f(u)$ such that it has an isolated fixed point at the origin, and the linear part $df(0)$ has one non-zero and two pure imaginary eigenvalues. We investigate the local flow on a neighborhood of the origin on the local center manifold and we distinguish between whether the origin is a center or a focus.

The problem of determining whether a system of differential equations $\dot{u} = f(u)$ in $\mathbb{R}^2$ having an isolated fixed point at the origin at which the linear part $df(0)$ has two pure imaginary eigenvalues is the well-known center-focus problem and is one of the most famous problems in qualitative theory of ordinary differential equations. It goes back to Poincaré. This problem is very hard and it has been intensively investigated by many authors (see for instance, [7,8,11,15–19] and the references therein).

In this paper we will focus on $\mathbb{R}^3$. Unlike the planar case, very little is known in the case of providing center conditions in dimensions greater than two. Very recent results in this direction are the papers [5,12,14]. In the first paper, the authors provide the basis to solve this problem by showing that the set of systems having a center on the center manifold at the origin correspond to a variety in the space of admissible coefficients. In all such very nice papers the authors apply these techniques to several families of systems with quadratic or cubic higher order terms.

The first class of systems that we will focus on are systems of the form

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + \alpha_1 u^2 + \alpha_2 uv + \alpha_3 uw + \alpha_4 v^2 + \alpha_5 vw + \alpha_6 w^2, \\ \dot{w} &= -w + c_1 u^2 + c_2 uv + c_3 v^2 \end{aligned} \quad (1)$$

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where $\alpha_1, \ldots, \alpha_6, c_1, c_2, c_3$ are parameters. A simple consequence of the Center Manifold Theorem (see [2]) shows that system (1) admits a local center manifold at the origin. We will write the center manifold as $w = \phi(u, v)$ for some smooth function ϕ. In this paper we study the center problem on a local center manifold of system (1).

We are able to obtain the following necessary and sufficient conditions for the existence of a center on the local center manifold for the following three six-parameter families of system (1):

(a) when $\alpha_3 = \alpha_5 = \alpha_6 = 0$;
(b) when $\alpha_2 = \alpha_4 = \alpha_5 = 0$;
(c) when $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Theorem 1. *System (1) with $\alpha_3 = \alpha_5 = \alpha_6 = 0$ admits a center on the local center manifold if and only if one of the following two conditions holds:*

(a.1) $\alpha_3 = -\alpha_1$;
(a.2) $\alpha_2 = 0$.

The proof of Theorem 1 is given in Section 3.

Theorem 2. *System (1) with $\alpha_2 = \alpha_4 = \alpha_5 = 0$ admits a center on the local center manifold if and only if one of the following two conditions holds:*

(b.1) $\alpha_3 = \alpha_6 = 0$;
(b.2) $c_3 = 2c_1$.

The proof of Theorem 2 is given in Section 4.

Theorem 3. *System (1) with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ admits a center on the local center manifold if and only if one of the following two conditions holds:*

(c.1) $\alpha_5 = \alpha_6 = 0$;
(c.2) $\alpha_5 = c_3 = 2c_1 - c_2$.
(c.3) $c_1 = c_2 = c_3 = 0$.

The proof of Theorem 3 is given in Section 5.

Another problem that we study in this paper is a conjecture posed in [14] related to the so-called Moon–Rand systems. The Moon–Rand systems developed by Moon and Rand in [13] to model control of flexible space structures, are systems of differential equations in $\mathbb{R}^3$ with polynomial right hand sides that have an isolated singularity at the origin at which the linear part has one negative and one pair of purely imaginary eigenvalues for all choices of the parameters.

The system is the following

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u - uw, \\ \dot{w} &= -\lambda w + f(u, v) \end{aligned} \tag{2}$$

where $f(u, v)$ is a polynomial or a rational function in the variables u, v. In [14] the authors conjectured that in the case of f being a homogeneous polynomial of degree n, called f_n, then system (2) has a center on the local center manifold at the origin if and only if

$$f_n(u, v) = cu^n + \frac{n}{\lambda} cu^{n-1} v.$$

They proved for the case $\lambda \in (0, \infty)$ and $f = f_2$, i.e., f being a homogeneous polynomial of degree two and also in the case $\lambda = 1$ and $f = f_3$, that is, f being a homogeneous polynomial of degree three. In this paper we prove the conjecture for the case $\lambda = 2$ and $f = f_3$ a homogeneous polynomial of degree three. Due to the enormous computation restrictions we will not consider the general case of $\lambda \in (0, \infty)$ and $f = f_3$. We recall that the case $f = f_3$ is computationally much harder than the case $f = f_2$ due to the presence of an extra parameter. Moreover, the parameter λ appears in the denominator of the focus quantities and growing λ brings computational difficulties. Finally, we want to mention that going from a fixed value of λ to considering it as a parameter provides again enormous computation difficulties and that is why at this moment we cannot consider the general case $f = f_3$ and $\lambda \in (0, \infty)$. More concretely we prove the following result.

Theorem 4. *System (2) with $\lambda = 2$ and f a homogeneous polynomial of degree three has a center on the center manifold at the origin if and only if $f = c_1 u^3 + \frac{3}{2} c_1 u^2 v$.*