Applied Mathematics and Computation 251 (2015) 203–210

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# The unified group classification method for the generalized nonlinear wave equation and its partial difference schemes ☆

Hanze Liu *

*School of Mathematical Sciences, Liaocheng University, Liaocheng, Shandong 252059, China*
*Department of Mathematics, Binzhou University, Binzhou, Shandong 256603, China*

ARTICLE INFO

*Keywords:*
Nonlinear second-order system
Discrete system
Lie group classification
Point symmetry
Partial difference scheme
Lattice

ABSTRACT

This paper is concerned with the generalized nonlinear second-order equation and its discrete counterpart furnished with different lattices, respectively. By the unified symmetry analysis method, the complete group classifications of the equations are performed. In the sense of point symmetry, all of the vector fields of the continuous nonlinear equation are obtained. As its special cases, the vector fields of some other important nonlinear equations are provided. Then, we develop the unified group classification method for dealing with partial difference schemes (PΔS), all of the point symmetries of the difference schemes are presented with respect to the given arbitrary analytic function $f(u)$.



## 1. Introduction

In [1], we studied the group classifications of the so-called Klein–Gordon wave equation $u_{xt} = p(u)$ and its difference schemes furnished with different lattices. All of the point symmetries of the equation are presented by the complete group classification method. In this paper, we consider the generalized second-order nonlinear wave equation as follows:

$$u_t = auu_x + bu_{xx} + f(u), \tag{1.1}$$

where $u = u(x,t)$ is the unknown function with respect to the independent variables $x$ and $t$, which denote the space variable and time in physics and applications, the parameters $a$ and $b$ are arbitrary constants, $f = f(u)$ is a given arbitrary analytic function with respect to the dependent variable $u = u(x,t)$. In general, we assume that $b \neq 0$ and $f'(u) \neq 0$ throughout this paper.

Clearly, a corresponding partial difference equation of Eq. (1.1) is of the following form:

$$\frac{u_{m,n+1} - u_{m,n}}{t_{m,n+1} - t_{m,n}} = au_{m,n}\frac{u_{m+1,n} - u_{m,n}}{x_{m+1,n} - x_{m,n}} + b\frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{(x_{m+1,n} - x_{m,n})^2} + f(u_{m,n}), \tag{1.2}$$

where $u_{m,n} = u(x_{m,n}, t_{m,n})$ denotes the unknown function in the discrete form, $m,\ n \in \mathbb{Z}$. That is, the reference point $(x_{m,n}, t_{m,n})$ in a plane can be arbitrarily shifted to the left, or to the right, up and down. Generally speaking, a partial difference scheme (PΔS) involves two objects, a difference equation, such as (1.2) and a lattice. In what follows, we will investigate the complete

☆ This work is supported by the National Natural Science Foundation of China under Grant Nos. 11171041, 11401274, and the Doctorial Foundation of Liaocheng University under Grant No. 31805.
* Address: School of Mathematical Sciences, Liaocheng University, Liaocheng, Shandong 252059, China.
*E-mail address:* hzliumath@hotmail.com

http://dx.doi.org/10.1016/j.amc.2014.11.017
0096-3003/© 2014 Published by Elsevier Inc.

group classifications of the partial difference schemes which are composed of the discrete Eq. (1.2) and the lattices as follows, respectively:

$$\begin{aligned}&x_{m+1,n} - 2x_{m,n} + x_{m-1,n} = 0,\\ &t_{m,n+1} - 2t_{m,n} + t_{m,n-1} = 0,\\ &x_{m,n+1} - x_{m,n} = 0,\\ &t_{m+1,n} - t_{m,n} = 0\end{aligned} \qquad \text{(I)}$$

and

$$\begin{aligned}&x_{m+1,n} - x_{m,n} - h_1 = 0,\\ &t_{m,n+1} - t_{m,n} - h_2 = 0,\\ &x_{m,n+1} - x_{m,n} = 0,\\ &t_{m+1,n} - t_{m,n} = 0,\end{aligned} \qquad \text{(II)}$$

where $h_1$ and $h_2$ are arbitrary constants.

Under our assumption, Eq. (1.1) includes a lot of important partial differential equations (PDEs) as its special cases. For example, if $f = 0$, then Eq. (1.1) becomes the following celebrated Burgers' equation (BE)

$$u_t = auu_x + bu_{xx}. \qquad (1.3)$$

So, Eq. (1.1) is also called a generalized Burgers' equation (GBE) sometimes.

On the other hand, if $a = 0$, then Eq. (1.1) is the nonlinear heat equation as follows

$$u_t = bu_{xx} + f(u). \qquad (1.4)$$

In particular, if $f = 0$, then Eq. (1.4) reduces to the following classical heat equation

$$u_t = bu_{xx}. \qquad (1.5)$$

Similarly, we can give the discrete forms of these equations in terms of the difference Eq. (1.2), the details are omitted here.

These equations are of great importance in mathematical physics, nonlinear theory and fluid mechanics, etc. In practice, many physical, mechanical and engineering models can be depicted by such types of equations. Particularly, the discrete schemes play a significant role in numerical computation, numerical simulation, biology and social sciences, and so on [2,3]. In [4–6], we studied the Burgers' types of equations by Lie symmetry analysis method, the symmetries, Bäcklund transformations (BTs) and exact solutions to the equations are investigated.

It is known that the symmetry approach is a systematic and powerful method for dealing with symmetries, integrability and exact solutions to differential and difference equations [1,2,4–16]. Moreover, the discrete equations serve as primary, fundamental mathematical models in physics and applications. The difference schemes are of more and more importance in mathematical physics, computer science, cellular theory and applications, neural nets, economics and social sciences, and so on. Especially, the difference schemes play a key role in numerical computation and simulation in nonlinear science and physical applications. However, most of the studies on symmetry, integrability and exact solutions to nonlinear systems are related to the continuous systems as far as we know. Compare to the continuous systems, the study of symmetries of discrete systems is a relatively recent subject, and it is more complicated than the former.

Furthermore, we note that the term $f(u)$ in the above equations is a composite function with respect to the given arbitrary function $f = f(u)$ and the unknown function $u = u(x,t)$, and this term affects the properties of the systems greatly, such as the symmetries, integrability and exact solutions.

The main purpose of this paper is to develop the unified group classification method for dealing with complete symmetry classifications of the generalized nonlinear second-order equation (1.1) and the partial difference schemes (1.2)+(I) and (1.2)+(II). The remainder of this paper is organized as follows: in Section 2, we perform Lie group classification on the generalized nonlinear second-order Eq. (1.1), and present all of the geometric vector fields of the equation, so the vector fields of the other important nonlinear equations are provided as its examples. In Section 3, the symmetry classifications of partial difference schemes are performed, all of the point symmetries of the difference schemes (1.2)+(I) and (1.2)+(II) are obtained by the unified symmetry analysis method. Finally, we summarize the findings and give some remarks in Section 4.

**Remark 1.1.** It is worth noting that Eq. (1.1) itself implies $f(0) = 0$, that is, the function $f = f(u)$ does not admit any nonzero constant term. More importantly, the difference Eq. (1.2) has Eq. (1.1) as its continuous limit. This is necessary for the discrete system in general, since it involves the convergence and stability of the difference scheme.

## 2. Lie group classification of Eq. (1.1)

In this section, we perform complete group classification on the nonlinear wave Eq. (1.1), then all of the geometric vector fields of the other equations such as Eqs. (1.3) and (1.4) are provided successively.