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New solutions for solving Boussinesq equation via potential symmetries method

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ABSTRACT

This work deals with the Boussinesq equation that describes the propagation of the solitary waves with small amplitude on the surface of shallow water. Firstly, the equation is written in a conserved form, a potential function is then assumed reducing it to a system of partial differential equations. The Lie-group method has been applied for determining symmetry reductions of the system of partial differential equations. The solution of the problem by means of Lie-group method reduces the number of independent variables in the given partial differential equation by one leading to nonlinear ordinary differential equations. The resulting non-linear ordinary differential equations are then solved numerically using MATLAP package.

1. Introduction

The Boussinesq equation is widely used in coastal and ocean engineering. One example among others is tsunami wave modeling. These equations can also be used to model tidal oscillations. Of course, these types of wave motion are perfectly described by the Navier–Stokes equations, but currently it is impossible to solve fully three-dimensional (3D) models in any significant domain. Thus, approximate models such as the Boussinesq equations must be used. Several members of the Boussinesq system have been studied in the past, including the classical Boussinesq system.

The existence of solutions for the Boussinesq system of equations had been obtained by Schonbek [1]. The exact solution of the classical Boussinesq equation had been presented by Krishan [2].

The numerical solution of the good Boussinesq equation had been expressed by Manornajan et al. [3] by using Galerkin Methods.

An exact traveling-wave solution of Boussinesq systems was presented by Chen [4]. It was found that it is sufficient to find a solution of an ordinary differential equation and by solving a system of nonlinear algebraic equation it was found the solution of the ordinary differential equation in a prescribed form.

Prabir et al. [5] derived a class of model equation that described the bi-directional propagation of small amplitude long waves on the surface of shallow water. The traveling solitary wave solutions are explicitly constructed for a class of lower order Boussinesq equation of higher-order. The existence and uniqueness of the solution to the Cauchy problem for a class of Boussinesq equation had been investigated by Wang et al. [6].

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A set of two-dimensional Boussinesq equations had been presented by Peregrine [7] which is the basis of much of the modern-day work. The model is invalid in case of deeper water rendering as their linearised dispersion characteristics rapidly diverge from the true behavior. This system of equations is limited to very shallow water.

Beji and Nadaoka [8] have produced an extended Boussinesq system, valid in a variable depth environment, by a simple algebraic manipulation of Peregrine's original system.

Bruzón and Gandarias [9] made a full analysis of a family of Boussinesq equations which include nonlinear dispersion by using the classical Lie method of infinitesimals. In their paper, they considered traveling wave reductions and presented some explicit solutions: solitons and compactons. They derived nonclassical and potential symmetries and proved that the nonclassical method applied to these equations leads to new symmetries, which cannot be obtained by Lie classical method.

Clarkson and Priestley [10] classified the generalized Boussinesq equation by the types of classical and nonclassical symmetry reduction it possesses. They studied the ordinary differential equation arising from their symmetry reductions to see whether they are of Painlevé -type. By virtue of the Painlevé conjecture they concluded that only the Boussinesq equation in the class of equations they have studied may be solvable by inverse scattering.

Some new similarity solutions of the modified Boussinesq equation are presented by Clarkson [11]. These new similarity solutions include reductions to the second and fourth Painlevé equations which are not obtainable using the standard Lie group method for finding group-invariant solutions of partial differential equations; they are determined using a new and direct method which involves no group theoretical techniques.

Clarkson and Ludlow [12] presented a new nonclassical symmetry reductions and exact solutions for a generalized Boussinesq equation. These symmetry reductions are obtained using the direct method, originally developed by Clarkson and Kruskal to study symmetry reductions of the Boussinesq equation, which involves no group theoretic techniques, and using these reductions, they obtained exact solutions expressible in terms of solutions of the second and fourth Painlevé equations, Jacobi, weierstrass elliptic functions and elementary functions.

Gandarias and Bruzón [13] applied the Lie-group formalism and the nonclassical method due to Bluman and Cole to deduce symmetries of the generalized Boussinesq equation, which has the classical Boussinesq equation as a special case. They studied the class of functions for which these equations admit either the classical or the nonclassical method. The reductions obtained are derived. Some new exact solutions are derived.

Kiraz [14] studied generalized Boussinesq equation reduced to previously unknown target ordinary differential equation by applying the extended Lie group transformation and similarity reduction. He obtained target ordinary equation and used it to find the exact solution of generalized Boussinesq equation.

Lockington et al. [15] concluded that, the similarity transforms of the Boussinesq equation in a semi-infinite medium are available when the boundary conditions are a power of time and the Boussinesq equation is reduced from a partial differential equation to a boundary-value problem.

Schäffer and Madsen [16] have suggested that a further set of extended Boussinesq equations involving an additional free parameter can be derived from [17] and these equation systems are all equivalent.

Peregrine [18] presented probably the first finite difference method for a Boussinesq-type equation system. However Abbot et al. [19,20] presented the finite difference solution of the original Boussinesq system for practical engineering problems, and to ensure accurate solutions they analyzed the methods carefully.

Clarkson and Kruskal [21] studied some new similarity reductions of the Boussinesq equation. These new similarity reductions, including some new reductions to the first, second, and fourth Painlevé equations, are determined using a new and direct method that involves no group theoretical techniques.

Levit and Winternitz [22] showed how a specific class of conditional symmetries can be used to reduce a partial differential equation to an ordinary one. In particular, for the Boussinesq equation, these conditional symmetries, together with the ordinary ones, provide all possible reductions to ordinary differential equations.

In this work, potential symmetries method is applied to the Boussinesq equation for determining symmetry reductions of partial differential equation, [23–28]. The resulting system of nonlinear differential equations is then solved using MATLAP package.

2. Mathematical formulation of the problem

The Boussinesq equation for the propagation of solitary waves with small amplitude on the surface of shallow water [29] is given by:

$$\frac{\partial^2 \bar{u}}{\partial t^2} + \left(\bar{u}\frac{\partial \bar{u}}{\partial x}\right)_x + \frac{\partial^4 \bar{u}}{\partial t^2} = 0, \tag{2.1}$$

where $\bar{u}(x,t)$ is the solitary wave velocity, x is the horizontal distance and t is the time.

The initial conditions are

$$\text{(i)}\quad \bar{u}(x,0) = f(x), \tag{2.2}$$

$$\text{(ii)}\quad \frac{\partial \bar{u}(x,0)}{\partial t} = 0. \tag{2.3}$$