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Bifurcations of a non-gravitational interaction problem

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ABSTRACT

This paper studies a particular planar problem which can be related to the interaction of two bodies under the action of a *non-gravitational force field*. Specifically, the mathematical description of this problem is a system of ordinary differential equations depending on a parameter ν. We investigate the topological structure of this system as ν varies along the entire real line $\mathbb{R}$. As we shall see there are five distinct cases for all $\nu \in \mathbb{R}$. A bifurcation scenario occurs at $\nu = 0$ and $\nu = 2$, respectively. Our study partly recovers the results known in Mcgehee (1981).

1. Introduction

The bifurcation analysis presented in this work is motivated by the classic planar two body problem, which studies the motion of two point particles that interact only with each other and that move in a fixed plane. For example, a single planet orbits around the sun, cf. [2,3].

A complete study of the solution behavior of such a problem is given in [3]. Then it is natural to ask: how do the solutions behave when the interaction force between these two bodies is non-gravitational? A particular case is that of two particles of unit mass moving in the plane $\mathbb{R}^2$. We fix particle A at the origin. Then the motion of particle B is described by [5]

$$\ddot{x} = F(x) = -\frac{\nu}{|x|^{\nu+2}}x, \tag{1}$$

where $\nu > 0$ is a parameter, $x \in \mathbb{R}^2$ denotes the position of particle B. It is shown in [5] that the topological structure of (1) changes dramatically at $\nu = 2$. However, the dynamics for $\nu < 0$ is not studied in [5].

The aim of this work is to investigate the topological structure of (1) for all $\nu \in \mathbb{R}$ from the mathematical point of view. We provide a complete bifurcation analysis, partly recovering the result in [5]. The methodology used is based on the blow up technique. We also provide a full interpretation of the blown up dynamics and its correspondence with the original flow.

Since the central force field F is conservative, it is straightforward to check that the potential energy function of the particle B is $U(x) = -\frac{1}{|x|^{\nu}}$. Introducing the vector $v = \dot{x}$, the velocity of the particle B, Eq. (1) can be written as a system of two first order ordinary differential equations:

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= -\frac{\nu}{|x|^{\nu+2}}x. \end{cases} \tag{2}$$

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Clearly, this is a 4-dimensional Hamiltonian system with Hamiltonian function

$$H = \frac{1}{2}|v|^2 - \frac{1}{|x|^\nu}, \tag{3}$$

which represents the total energy of the particle B, and is a conserved quantity, i.e., H is constant along any solution curve of system (2).

2. Blow up of the singularity

Observe that (1), or equivalently (2), is not well defined at the origin $x = 0$. Physically, at $x = 0$ the two particles collide. The potential energy goes to $-\infty$ as $x \to 0$. In order to study the orbits near the singular point $x = 0$, we perform Mcgehee's blow up [4,5]. To do this, we rewrite the vector v in the new variables (v_r, v_θ) in the following way:

$$v = v_r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + v_\theta\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{4}$$

Meanwhile, we introduce the polar coordinates (r, θ) with $x = (r\cos\theta, r\sin\theta)$, where $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$. Observe that the point $x = 0$ is mapped to $r = 0$ under the blow up transformation. By straightforward computations system (1) can be written in the variables $(r, \theta, v_r, v_\theta)$ as

$$\begin{cases} \dot{r} &= v_r, \\ \dot{\theta} &= \frac{v_\theta}{r}, \\ \dot{v}_r &= -\frac{\nu}{r^{\nu+1}} + \frac{v_\theta^2}{r}, \\ \dot{v}_\theta &= -\frac{v_r v_\theta}{r}. \end{cases} \tag{5}$$

However, (5) has still a singularity at $r = 0$. To overcome such a situation, we introduce the scaled variables (u_r, u_θ) with $u_r = r^{\frac{\nu}{2}} v_r,\ u_\theta = r^{\frac{\nu}{2}} v_\theta$, and parameterize the time by τ, where $t = r^{\frac{\nu}{2}+1}\tau$. Then the system (5) can be written in the coordinates $(r, \theta, u_r, u_\theta)$ as

$$\begin{cases} r' &= r u_r, \\ \theta' &= u_\theta, \\ u_r' &= \frac{\nu}{2}u_r^2 + u_\theta^2 - \nu, \\ u_\theta' &= \left(\frac{\nu}{2} - 1\right)u_r u_\theta, \end{cases} \tag{6}$$

where $'$ denotes derivative with respect to the re-scaled time τ. Observe that now (6) is a complete smooth vector field in $\mathbb{R}^4$. The subspace $\{r = 0\}$ is invariant. That implies that the solution curves of (6) restricted to $\{r = 0\}$ arrange the whole dynamics for sufficiently small $r > 0$. The equilibrium set of $(6)|_{r=0}$ is

$$\Lambda = \{r = 0, u_\theta = 0, u_r = \pm\sqrt{2}\}. \tag{7}$$

Remark 2.1.

- A *collision orbit* is a trajectory of (1) whose limit is $x = 0$. Such a behavior is equivalent to an orbit of the regularized system (6) which approaches Λ. See more details below in Lemma 3.1.
- The vector fields (5) and (6) are topologically equivalent [1] for $r > 0$. On the other hand, $\{r = 0\}$ is an invariant space of (6), and therefore, the flow in such invariant space arranges the dynamics for $r > 0$ but sufficiently small. That is, we can obtain a qualitative description of the orbits of (5) by studying (6) restricted to $r = 0$.

3. Bifurcations and dynamics

Note that the last two equations of (6) are independent of r and θ and play a key role in determining the dynamics of the entire system. Therefore, we investigate these two equations as a separate system. Since we are interested in the topological change of (6) as $\nu \in \mathbb{R}$ varies we regard ν as an extra parameter and consider the following extended system

$$\begin{cases} u_r' = \frac{\nu}{2}u_r^2 + u_\theta^2 - \nu, \\ u_\theta' = \left(\frac{\nu}{2} - 1\right)u_r u_\theta, \\ \nu' = 0. \end{cases} \tag{8}$$