Applied Mathematics and Computation 251 (2015) 300–317

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

First and second-order optimality conditions for nonsmooth vector optimization using set-valued directional derivatives

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ARTICLE INFO

Keywords:
Nonsmooth vector optimization
Optimality condition
Weak solution
Firm solution
Set-valued directional derivative

ABSTRACT

We investigate a nonsmooth vector optimization problem with a feasible set defined by a generalized inequality constraint, an equality constraint and a set constraint. Both necessary and sufficient optimality conditions of first and second-order for weak solutions and firm solutions are established in terms of Fritz-John–Lagrange multiplier rules using set-valued directional derivatives and tangent cones and second-order tangent sets. We impose steadiness and strict differentiability for first and second-order necessary conditions, respectively; stability and l-stability for first and second-order sufficient conditions, respectively. The obtained results improve or include some recent known ones. Several illustrative examples are also provided.

1. Introduction

In the last years, first and second-order optimality conditions for optimization problems have been investigated intensively in numerous publications (see [1–28] and references therein). In nonsmooth optimization, the major approach to optimality conditions is using generalized derivatives to replace Fréchet and Gâteaux derivatives which do not exist. Several kinds of derivatives have been utilized such as pseudo Jacobian and Hessian [12], second-order subdifferentials [7], first and second-order approximations [6,18], directional derivatives [1,24,26], Neustadt derivatives [21,22] and set-valued directional derivatives [2,3,8,9,11,15,17,19,20]. This paper is concerned with necessary and sufficient optimality conditions of first and second-order for possibly infinite-dimensional nonsmooth vector optimization, using set-valued directional derivatives as generalized derivatives. The optimization problem under our consideration is

$$\min f(x), \text{s.t.} x \in S, \quad g(x) \in -K, h(x) = 0, \tag{P}$$

where $f : X \to Y, g : X \to Z$, and $h : X \to W$ are mappings, X and W Banach spaces, Y and Z normed spaces, $S \subset X, C \subset Y$ a closed convex cone, and $K \subset Z$ a convex set.

First-order and second-order optimality conditions for (P) in terms of pseudo Jacobian and Hessian matrices of continuous and continuously differentiable functions, respectively (shortly resp), were established by Luc–Jeyakumar [12]. In [15], Jiménez–Novo used the notion of (first-order) contingent derivatives of single-valued maps to develop first-order optimality conditions for (P) under steadiness or stability assumptions of data in finite dimensional spaces. Second-order smooth optimality conditions with the envelope-like effect for (P) (where $Y = \mathbb{R}$) via a Kuhn-Tucker–Lagrange multiplier rule were given by Kawasaki [16]. His results were developed by various authors in [4,5,23,25,26], considering always C^2 scalar programs and by Maruyama in [22], studying nonsmooth scalar problem (P) and using the notion of second-order Neustadt derivatives. In vector programming, the first result of this type was given in [13,14] for smooth cases. In finite-dimensional nonsmooth multiobjective programming, Gutiérrez–Jiménez-Novo [11] used set-valued second-order directional derivatives to establish second-order optimality conditions with the envelope-like effect. They considered Fréchet differentiable functions whose

http://dx.doi.org/10.1016/j.amc.2014.11.061
0096-3003/

Fréchet derivative is continuous or stable at the point of study. Their results were generalized by Khanh–Tuan in [18,19], using approximations and set-valued directional derivatives, resp, under relaxed assumptions of data: (strict) differentiability or l-stability. Motivated by the works reported in [4,5,11–16,18,19,22,23,25,26], in this note we develop first-order necessary and sufficient optimality conditions and second-order ones with the envelope-like effect for problem (P) in possibly infinite-dimensional normed spaces by using set-valued directional derivatives and tangent cones and second-order tangent sets. For the first time, to our knowledge, we employ (second-order) asymptotic directional derivatives of single-valued vector functions. We impose steadiness and strict differentiability for first and second-order necessary conditions, resp; stability and l-stability for first and second-order sufficient conditions, resp. The obtained results improve or include some known ones in scalar or vector optimization (see for instance [7,11,15,16,33]).

The organization of the paper is as follows. In Section 2 we recall some preliminary facts, including those concerning l-stable functions and second-order tangency. Section 3 is devoted to set-valued directional derivatives and properties which will be in use. In Section 4, we establish first and second-order necessary optimality conditions for local weak solutions of (P). Section 5 contains first and second-order sufficient optimality conditions for local firm solutions.

2. Mathematical preliminaries and auxiliary results

Our notations are basically standard. $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ and $\mathbb{R}$ is the set of real numbers. For a normed space X, X^* stands for the topological dual of X; $\langle ., . \rangle$ is the canonical pairing. $\|.\|$ is used for the norm in any normed space (from the context no confusion occurs). $d(y, S)$ denotes the distance from a point y to a set S. $B_n(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$, $S_n = \{y \in \mathbb{R}^n : \|y\| = 1\}$, $B_X(x, r) = \{y \in X : \|x - y\| < r\}$ and $S_X = \{y \in X : \|y\| = 1\}$. $B(X, X)$ is the space of bounded bilinear mappings from $X \times X$ into $\mathbb{R}$. For a cone $C \subset X$, $C^* = \{c^* \in X^* : \langle c^*, c \rangle \geqslant 0, \forall c \in C\}$ is the positive polar cone of C. For $A \subset X$, int$A$, cl$A$, bd$A$, cone$A$, co$A$, and $A(x)$ stand for the interior, closure, boundary, conical hull, convex hull of A and, conical hull of the translate $A + x$, resp. For $t > 0$ and $k \in \mathbb{N}$, $o(t^k)$ designates a point in a considered space (which is clear from the context) depending on t such that $o(t^k)/t^k \to 0$ as $t \to 0^+$. $C^{1,1}$ is used for the space of Fréchet differentiable mappings whose Fréchet derivative is locally Lipschitz.

Now we recall the notions of tangent cones and second-order tangent sets that we will use later. Let X be a normed space.

Definition 2.1. Let $x_0, u \in X$, $S \subset X$ and $\gamma \in \{0, 1\}$.

(a) The contingent (or Bouligand) cone of S at x_0 is

$$T(S, x_0) = \{v \in X : \exists t_k \to 0^+, \exists v_k \to v, \forall k \in \mathbb{N}, x_0 + t_k v_k \in S\}.$$

(b) The interior tangent cone of S at x_0 is

$$IT(S, x_0) = \{v \in X : \forall t_k \to 0^+, \forall v_k \to v, \forall k \text{ large enough}, x_0 + t_k v_k \in S\}.$$

(c) The second-order contingent set of index γ of S at (x_0, u) is

$$T^2_\gamma(S, x_0, u) = \left\{w \in X : \exists (t_k, r_k) \to (0^+, 0^+) : \frac{t_k}{r_k} \to \gamma, \exists w_k \to w, \forall k \in \mathbb{N}, x_0 + t_k u + \frac{1}{2} t_k r_k w_k \in S\right\}.$$

(d) The second-order adjacent set of index γ of S at (x_0, u) is

$$A^2_\gamma(S, x_0, u) = \left\{w \in X : \forall (t_k, r_k) \to (0^+, 0^+) : \frac{t_k}{r_k} \to \gamma, \exists w_k \to w, \forall k \in \mathbb{N}, x_0 + t_k u + \frac{1}{2} t_k r_k w_k \in S\right\}.$$

(e) The second-order interior tangent set of index γ of S at (x_0, u) is

$$IT^2_\gamma(S, x_0, u) = \left\{w \in X : \forall (t_k, r_k) \to (0^+, 0^+) : \frac{t_k}{r_k} \to \gamma, \forall w_k \to w, \forall k \text{ large enough}, x_0 + t_k u + \frac{1}{2} t_k r_k w_k \in S\right\}.$$

When $\gamma = 0$, the sets $T^2_0(S, x_0, u)$, $A^2_0(S, x_0, u)$ and $IT^2_0(S, x_0, u)$ are cones and called the asymptotic second-order contingent cone, the asymptotic second-order adjacent cone and the asymptotic second-order interior tangent cone, resp. When $\gamma = 1$, the sets $T^2_1(S, x_0, u)$, $A^2_1(S, x_0, u)$ and $IT^2_1(S, x_0, u)$ are said to be the second-order contingent set, the second-order adjacent set and the second-order interior tangent set, resp. The cones $T(S, x_0)$ and $IT(S, x_0)$ and the sets $T^2_1(S, x_0, u)$, $A^2_1(S, x_0, u)$ and $IT^2_1(S, x_0, u)$ are well-known. The cones $A^2_0(S, x_0, u)$ and $T^2_0(S, x_0, u)$ were used by Penot [25,26]. The cone $IT^2_0(S, x_0, u)$ was introduced by Giorgi et al. [10]. For a systematics survey of second-order tangent sets and their application to vector optimization, see [10]. Note that if $x_0 \notin$ cl$S$, then all the above tangent sets are empty.

Some basic properties of the above first-order and second-order tangent sets are listed in the following proposition.

Proposition 2.1. *Let $x_0, u \in X$, $S \subset X$ and $\gamma \in \{0, 1\}$. Then, the following are satisfied*

(i) $IT^2_\gamma(S, x_0, u) \subset A^2_\gamma(S, x_0, u) \subset T^2_\gamma(S, x_0, u) \subset \text{clcone}[\text{cone}(S - x_0) - u]$;