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Penalty approach to a nonlinear obstacle problem governing American put option valuation under transaction costs

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ABSTRACT

We propose a penalty method for a finite-dimensional nonlinear complementarity problem (NCP) arising from the discretization of the infinite-dimensional free boundary/obstacle problem governing the valuation of American options under transaction costs. In this method, the NCP is approximated by a system of nonlinear equations containing a power penalty term. We show that the mapping involved in the system is continuous and strongly monotone. Thus, the unique solvability of both the NCP and the penalty equation and the exponential convergence of the solution to the penalty equation to that of the NCP are guaranteed by an existing theory. Numerical results will be presented to demonstrate the convergence rates and usefulness of this penalty method.

1. Introduction

Financial options are derivative instruments which can usually be traded in a secondary financial market. An option on a stock is a contract which gives its holder the right, not obligation, to sell (put option) or buy (call option) a certain number of the shares at the prescribed price called strike price. There are two major types of option: European option and American option. The former can only be exercised on a given future date (expiry date) while the latter can be exercised on or before the expiry date. Since an option can be traded in a financial market, a natural question is how to determine the value of the option at any time before the expiry date. In a complete market without transaction costs, Black–Scholes [3] proposed a partial differential equation pricing model for European options. However, when trading in bonds or/and stocks involves transaction cost, the Black–Scholes option pricing model does not hold anymore. To overcome this difficulty, various models have been proposed to price European options under transaction costs [19,4,23,2,15]. All these models give rise to a nonlinear Black–Scholes equation. There are also utility-maximization based models for determining the so-called reservation prices of European and American options under transaction costs [6–8,22]. These models are of the form of a set of Hamilton–Jacobi–Bellman equations for both European and American options.

The aforementioned models in the form of the nonlinear Black–Scholes equation can hardly be solvable analytically. In practice, approximate solutions to such a model are always sought. Several discretization schemes such as those in [9,1,14] have been proposed for solving this nonlinear Black–Scholes equation. In [20] we have proposed an upwind finite difference method for the equation and proved the convergence of the numerical scheme. However, it is known that the price

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of an American option is governed by a constrained optimization problem involving the nonlinear Black–Scholes operator and, to our best knowledge, how to price American put options of this type still remains an open problem.

In this work, we will present a computational technique for solving the infinite-dimensional nonlinear obstacle or complementarity problem (NCP), or equivalently a nonlinear variational inequality, governing American option valuation under transaction costs involving the nonlinear Black–Scholes operator developed for European option valuation in [2]. The infinite dimensional NCP is first discretized by the numerical scheme proposed and analyzed recently in [20] for the European option problem under transaction costs. A power penalty method is then proposed for the NCP in finite dimensions arising from the discretization. We show that the mapping defining the NCP is locally Lipschitz continuous and strongly monotone so that an existing convergence theory applies to our problem. Numerical results will then be presented to demonstrate the convergence rates and usefulness of the method. Although we only consider the nonlinear Black–Scholes operator in [2] for discussion clarity, the principle developed in this work applies to other nonlinear Black–Scholes' operators such as those in [19,4,23,15]. We comment that other methods such as those in [10,13] may also be used for solving the discretized nonlinear NCP.

For an American Vanilla put option without transaction cost in a complete market, its value $v(S,t)$ as a function of its underlying asset/stock price S and time t is governed by the following linear complementarity or obstacle problem [27] involving the Black–Scholes differential operator:

$$\mathcal{L}_0 v(S,t) := -\frac{\partial v}{\partial t} - \frac{1}{2}\sigma_0^2 S^2 \frac{\partial^2 v}{\partial S^2} - rS\frac{\partial v}{\partial S} + rv \geqslant 0, \tag{1}$$

$$v(S,t) - u^*(S) \geqslant 0, \tag{2}$$

$$\mathcal{L}_0 v(S,t)\cdot[v(S,t) - u^*(S)] = 0, \tag{3}$$

for $(S,t) \in (0, S_{\max}) \times [0,T)$ satisfying the payoff/terminal and boundary conditions

$$v(S,T) = u^*(S), \quad S \in (0, S_{\max}),$$
$$v(0,t) = u^*(0) = K, \quad t \in (0,T],$$
$$v(S_{\max},t) = u^*(S_{\max}) = 0, \quad t \in (0,T],$$

where K is the strike price of the option, $S_{\max} \gg K$ is a positive constant defining a computational upper bound on S, σ_0 is a constant volatility of the asset, $r > 0$ is a constant risk-free interest rate, and $u^*(S)$ is the payoff function given by

$$u^*(S) := \max(K - S, 0). \tag{4}$$

In the presence of transaction costs and under the transformation $\tau = T - t$, the value of an American Vanilla put option, $u(S,\tau)$ is governed by the following nonlinear complementarity problem:

$$\mathcal{L}(u)(S,\tau) := \frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2\left(\tau, S, \frac{\partial u}{\partial S}, \frac{\partial^2 u}{\partial S^2}\right) S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru \geqslant 0, \tag{5}$$

$$u(S,\tau) - u^*(S) \geqslant 0, \tag{6}$$

$$\mathcal{L}(u)(S,\tau)\cdot[u(S,\tau) - u^*(S)] = 0, \tag{7}$$

for $(S,\tau) \in (0, S_{\max}) \times [0,T)$, where σ is the modified volatility as a nonlinear function of $\tau, S, \frac{\partial u}{\partial S}$ and $\frac{\partial^2 u}{\partial S^2}$, and $u^*(S)$ is the payoff function defined in (4). The initial and boundary conditions become

$$u(S,0) = u^*(S), \quad S \in (0, S_{\max}), \tag{8}$$

$$u(0,\tau) = K, \quad \tau \in (0,T], \tag{9}$$

$$u(S_{\max},\tau) = 0, \quad \tau \in (0,T]. \tag{10}$$

We comment that different types of American options have different payoff functions. The most common one is the payoff function for Vanilla American put options defined in (4). For clarity, we only consider this type of payoff functions in this work and the results to be presented can also be used for other types of payoff functions.

Various models for the nonlinear volatility have been proposed, for example [19,4,15,18,2]. A notable one is the following nonlinear volatility model proposed by Barles and Soner [2]:

$$\sigma^2\left(\tau, S, \frac{\partial^2 u}{\partial S^2}\right) := \sigma^2\left(e^{r\tau} a^2 S^2 \frac{\partial^2 u}{\partial S^2}\right) = \sigma_0^2\left(1 + \Psi\left[e^{r\tau} a^2 S^2 \frac{\partial^2 u}{\partial S^2}\right]\right), \tag{11}$$

where $a = \kappa\sqrt{\nu N}$ with κ being the transaction cost parameter, ν a risk aversion factor and N the number of options to be sold. In the rest of this paper, we simply refer to a as the transaction parameter. The function Ψ is the solution to the following nonlinear initial value problem:

$$\Psi\prime(z) = \frac{\Psi(z) + 1}{2\sqrt{z\Psi(z)} - z} \quad \text{for} \quad z \neq 0 \quad \text{and} \quad \Psi(0) = 0. \tag{12}$$