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Extremal curves for weighted elastic energy in surfaces of 3-space forms

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Dedicated to Prof. B.-Y. Chen on the occasion of his 70th birthday.

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ABSTRACT

A variational problem closely related to the bending energy of curves contained is surfaces of real 3-space forms is considered. We seek curves in a surface which are critical for the elastic energy when this is weighted by the total squared normal curvature energy, under two different sets of constraints: clamped curves and one free end curves of constant length. We start by deriving the first variation formula and the corresponding Euler–Lagrange equations and natural boundary conditions of these energies and characterize critical geodesics. We show how surfaces locally foliated by critical geodesics can be found by using the fundamental theorem of submanifolds. In order to find explicit solutions we classify complete rotation surfaces in a real space form for which every parallel is critical.

1. Introduction

Extending the classical Bernoulli's model for curves in the plane, a curve $\gamma : I \to M^n$, isometrically immersed in a Riemannian manifold M^n, is said to be an *elastica* if it is a minimizer of the *total squared curvature*

$$\mathcal{F}(\gamma) = \int_\gamma \kappa_g^2 \, ds, \tag{1.1}$$

where κ_g denotes the *geodesic curvature* of γ. Thus, in Euclidean ambient spaces, the total squared curvature functional can be thought of as the *bending energy* of the curve and its equilibrium positions as mathematical models for *thin elastic rods*. More generally, immersed curves of a Riemannian manifold which are *critical* (not necessarily minima) for the total squared curvature (1.1) are usually called *elastic curves* or, simply, *elasticae* and, of course, there are different variational problems in Riemannian manifolds associated to the various constraints and boundary conditions that can be considered.

In 1743 Euler obtained the plane elastic curves by quadratures [8], whilst the first explicit parametrizations of the Euler elasticae were given by Saalchütz in 1880. Stability of the plane elasticae was studied by Max Born in his Ph.D. thesis (1906) who, in addition, noticed that the slope of the elasticae satisfies the equation for the mathematical pendulum. Also at the beginning of the 20th century, Radon [22] and Irrgang [15] analyzed the *free elastic curves* in $\mathbb{R}^3$ (i.e. with no constraint on the length of the curve variation). In recent years, many mathematicians have shown interest on the subject again and, in particular, the elastica problem in Riemannian manifolds has come to the attention of many geometers who have investigated related problems using different approaches [1,5,13,14,16,17,19,20] (see also [10,11]) for recent surveys). As

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a consequence, elastica in real space forms are well known. Generalizations of the bending energy functional (1.1) to curvature energy functionals involving several Frenet curvatures of the curve, have interesting applications to several fields: in Physics to the construction of models of relativistic particles and p-branes; in Biophysics to the study of membranes and vesicles; and, in Mathematics to the Theory of Submanifolds, where they can be used, for instance, to construct algorithms which provide efficient ways to obtain new examples of Chen-Willmore submanifolds (see [2,10] for more details and references).

However, on the one hand, little is known for elasticae in non-constant curvature Riemannian manifolds and, on the other hand, the Euler–Lagrange equations for these generalized elastic energy functionals do not appear to be tractable in general. In order to make these problems more accessible, we consider curves which are isometrically immersed in a surface of a 3-pace form. For these curves, in addition to the geodesic curvature (which determines the bending energy), one has another important geometric quantity which is defined on them due to the fact that they live on the surface: the *normal curvature*. According to [23], minimizers of the total squared normal curvature $\int \kappa_n^2$ appear to have some significance in the self-assembly analysis of thin films formed by block copolymers in a cylindrical phase. The purpose of this paper is to study the variational problem associated to the energy $\mathcal{F}_{\nu\mu}$ as given in (3.1) for curves isometrically immersed in a surface of a 3-pace form. That is, we consider the case of curves, isometrically immersed in surfaces, whose shape is controlled by the bending energy (1.1) when this is weighted by the total squared normal curvature of the curve.

In Section 3 we consider two variational problems associated to the energy $\mathcal{F}_{\nu\mu}$ for curves γ contained in a surface of real 3-space form $M^2 \subset M^3(c),\ c = 0,\ 1,\ -1$. We first take variations of γ by curves of M^2 with the same initial point and initial velocity, same length and arbitrary endpoint (in short, *free end problem*). Then, we analyze the problem for variations of *curves with clamped ends* (that is, curves with prescribed zero and first order boundary data). The Euler–Lagrange equations are easily obtained combining results in [19,4], since the total squared normal curvature is a special case of the energy considered in the latter. Moreover, the natural boundary conditions might be obtained by using techniques similar to those in [19,4]. Rather than doing this, here we use a little trick which makes the computation shorter and characterize critical geodesics in both cases. In Section 4 we study the special case $\nu = \mu$. Now, $\mathcal{F}_{\nu\mu}$ is nothing but the elastic energy of the curve in $M^3(c)$, but the curve variations are taken in the surface containing it. This problem will be referred to as the *surface constrained problem for the elastic energy* $\mathcal{F}_{\nu\mu}$. The surface constrained problem for Euclidean surfaces was first posed and studied by Santaló, [24]. If $c = 0$, that is for surfaces of $\mathbb{R}^3$, the Euler–Lagrange equations of the surface constrained elastic energy were obtained in [24] (clamped case) and in [21] (free end case). By using a different approach, we analyze here all cases $c = 0,\ 1,\ -1$ simultaneously, paying special attention to umbilical surfaces. We also construct surfaces locally foliated by critical geodesics. Finally, although geodesics of surfaces in $M^3(c)$ need not be critical curves necessarily, we observe that, as a consequence of our previous results, rotation surfaces of $M^3(c)$ are locally foliated by critical meridians. In connection with this fact, in Section 5 we classify complete rotation surfaces of $M^3(c)$ for which every parallel is critical for $\mathcal{F}_{\nu\mu}$.

2. Preliminaries

Let $M^3(c),\ c \in \{\pm 1, 0\}$, be the simply connected space form of sectional curvature c with metric $\langle , \rangle$, and consider an orientable surface M^2 isometrically immersed in $M^3(c)$ (we assume that either M^2 is orientable or that the curves under consideration lie within a coordinate neighborhood). As usual, $\widetilde{\nabla}$ and ∇ will denote the Levi–Civita connections of $M^3(c)$ and M^2, respectively, and we will use R for the curvature tensor of $M^3(c)$.

Along this paper, the symbol $\mathbb{I}$ will be used to denote the interval $[0, 1]$. Let $\beta : \mathbb{I} \to M^3(c)$ be a smooth immersed curve $\beta(t)$, and denote by $\gamma(s)$ its unit speed reparametrization with unit tangent $T(s) = \gamma'(s)$, where $'$ denotes derivative with respect to the arc-length parameter s. We say that $\beta(t)$ is of *rank* 0 if $\gamma(s)$ is a geodesic. If $\gamma'(s)$ and $\widetilde{\nabla}_{\frac{\partial}{\partial s}}\gamma'(s)$ are everywhere linearly independent, we will say that $\beta(t)$ is of *rank* 1. In such a case the unit *normal* and *binormal* vector fields on $\gamma(s)$ are defined by $N(s) = \widetilde{\nabla}_{\frac{\partial}{\partial s}}\gamma'(s) \Big/ \left\| \widetilde{\nabla}_{\frac{\partial}{\partial s}}\gamma'(s) \right\|$ and $B(s) = T(s) \times N(s)$, respectively, where $\times$ is the vector product on $M^3(c)$. Thus, $\{T, N, B\}$ represents the usual *Frenet frame* on γ whilst the *(Frenet) curvature* and *torsion* functions of γ in $M^3(c)$ $\{\kappa(s), \tau(s)\}$ are defined by the following *Frenet equations*

$$
\begin{aligned}
\widetilde{\nabla}_T T &= \kappa \cdot N, \\
\widetilde{\nabla}_T N &= -\kappa \cdot T + \tau \cdot B, \\
\widetilde{\nabla}_T B &= -\tau \cdot N.
\end{aligned}
\tag{2.1}
$$

Assume now that which $\gamma(s)$ is contained in the surface $M^2 \subset M^3(c)$ and let ξ be a choice of the unit normal vector to M^2. If we consider the *Darboux frame* $\{T, \eta, \xi\}$, where $\eta := \xi \times T$. Then, we can write

$$
\begin{aligned}
N &= \cos\psi \cdot \xi + \sin\psi \cdot \eta, \\
B &= \sin\psi \cdot \xi - \cos\psi \cdot \eta,
\end{aligned}
\tag{2.2}
$$

where ψ is the angle which N makes with the normal to the surface ξ. Moreover, the *geodesic curvature*, the *normal curvature* and the *geodesic torsion* of γ in M^2, denoted, respectively, by $\{\kappa_g, \kappa_n, \tau_g\}$, are defined by the following *Darboux's equations*