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Some numerical methods for solving nonlinear equations by using decomposition technique

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ABSTRACT

In this paper, we use the system of coupled equations involving an auxiliary function together with decomposition technique to suggest and analyze some new classes of iterative methods for solving nonlinear equations. These new methods include the Halley method and its variant forms as special cases. Various numerical examples are given to illustrate the efficiency and performance of the new methods. These new iterative methods may be viewed as an addition and generalization of the existing methods for solving nonlinear equations.

1. Introduction

It is well known that a wide class of problems, which arises in diverse disciplines of mathematical and engineering science can be studied by the nonlinear equation of the form $f(x) = 0$. Numerical methods for finding the approximate solutions of the nonlinear equation are being developed by using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques, see [1–17] and the references therein.

In this paper, we use alternative decomposition technique to suggest the main iterative schemes which generates the iterative methods of higher order. First of all, we rewrite the given nonlinear equation along with the auxiliary function as an equivalent coupled system of equations using the Taylor series. This approach enables us to express the given nonlinear equation as sum of linear and nonlinear equations. This way of writing the given equation is known as the decomposition and plays the central role in suggesting the iterative methods for solving nonlinear equations $f(x) = 0$.

In this work, we use the system of coupled equations to express the given nonlinear equations as a sum of linear and nonlinear operators involving the auxiliary function $g(x)$. This auxiliary function helps to deduce several iterative methods for solving nonlinear equations. The effectiveness and efficiency of the auxiliary function can be observed in the next section for deriving the robust iterative methods for solving nonlinear equations.

Results obtained in this paper, suggest that this new technique of decomposition is a promising tool. In section 2, we sketch out the main ideas of this technique and suggest some multi-step iterative methods for solving nonlinear equations. One can notice that if the derivative of the function vanishes, that is $|f'(x_n)| = 0$, during the iterative process, then the sequence generated by the Newton method or the methods derived in [1–17] are not defined. Due to cardinal sin of division results in a mathematical breakdown. This is another motivation of the paper that the derived higher order methods also converge even if the derivative vanishes during the iterative process. We also show that the new methods include Newton

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method and Halley method and their variant forms as special cases. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative methods. Our results can be considered as an important improvement and refinement of the previously known results.

2. Iterative methods

In this section, we suggest some new iterative methods for solving nonlinear equations by using decomposition technique involving an auxiliary function. This auxiliary function diversifies the main recurrence relations for the best implementation of the methods to obtain the approximate solution of nonlinear equations. This technique purposes main iterative schemes to provide higher order convergent iterative methods.

Consider the nonlinear equation of the type

$$f(x) = 0. \tag{1}$$

Assume that p is a simple root of nonlinear equation (1) and γ is an initial guess sufficiently close to p. Let $g(x)$ be the auxiliary function, such that

$$f(x)g(x) = 0. \tag{2}$$

We can rewrite the nonlinear equation (2) as a system of coupled equations by using Taylor series technique as:

$$f(\gamma)g(\gamma) + [f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)](x-\gamma) + h(x) = 0. \tag{3}$$

Equation (3) can be written in the following form

$$h(x) = f(x)g(\gamma) - f(\gamma)g(\gamma) - [f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)](x-\gamma). \tag{4}$$

where γ is the initial approximation for a zero of (1). We can rewrite equation (4) in the following form

$$x = \gamma - \frac{f(\gamma)g(\gamma)}{[f(\gamma)g'(\gamma) + f'(\gamma)g(\gamma)]} - \frac{h(x)}{[f(\gamma)g'(\gamma) + f'(\gamma)g(\gamma)]}. \tag{5}$$

We express (5), in the following form as:

$$x = c + N(x), \tag{6}$$

where

$$c = \gamma - \frac{f(\gamma)g(\gamma)}{[f(\gamma)g'(\gamma) + f'(\gamma)g(\gamma)]}, \tag{7}$$

and

$$N(x) = -\frac{h(x)}{[f(\gamma)g'(\gamma) + f'(\gamma)g(\gamma)]}. \tag{8}$$

Here $N(x)$ is a nonlinear function.

We now construct a sequence of higher order iterative methods by using the following decomposition technique, which is mainly due to Daftardar-Gejji and Jafari [5]. This decomposition of the nonlinear function $N(x)$ is quite different from that of Adomian decomposition.

The main idea of this technique is to look for a solution having the series form

$$x = \sum_{i=0}^{\infty} x_i. \tag{9}$$

The nonlinear operator N can be decomposed as

$$N(x) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right) \right\}. \tag{10}$$

Combining (7), (9) and (10), we have

$$\sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right) \right\}. \tag{11}$$