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Numerical algorithm to solve system of nonlinear fractional differential equations based on wavelets method and the error analysis

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ABSTRACT

In this paper, a system of nonlinear fractional differential equations (FDEs) are considered. They have been solved by Legendre wavelets method combining with its operational matrix. However, there are no articles about solving this system using wavelets method. The main purpose of this technique is to transform the initial equations into a nonlinear system of algebraic equations which can be solved easily. The convergence and error analysis are presented to show the correctness and feasibility of method proposed for solving the above mentioned problem. Finally, the applicability and efficiency of the mentioned approach are demonstrated by three numerical examples.

1. Introduction

Fractional calculus has attracted increasing attention for decades since it plays a vital role in different disciplines of science and engineering [1–3]. Compared to integer order differential equation, fractional differential equation has the advantage that it can better describe some natural physics processes and dynamic system processes [4], because the fractional order differential operators are non-local operators. Many physics, chemistry and engineering systems can be elegantly modeled with the help of the FDEs, such as dielectric polarization [5], viscoelastic systems [6], control theory [7], chaotic behavior [8] and electrolyte–electrolyte polarization [9], and so on.

In the past several decades, many scholars have devoted themselves to studying such systems. Matignon, Tavazoei and Haeri investigated the stability results for linear fractional order systems in [10,11]. Daftardar-Gejji, Bonilla, Odibat et al. proposed an analytic study on linear systems of FDEs, and discussed the existence and uniqueness of solutions [12–14]. But most FDEs do not have exact solutions, so numerical and approximate techniques must be presented. Recently, numerical solutions for some classes of fractional order systems have been constructed by many authors for example homotopy analysis method [15], variational iteration method [16], finite difference method [17,18], collocation method [19–21], Adomian decomposition method [22,23], differential transform method [24].

Wavelets theory has been paid considerable attention from many scholars. It has been applied in a wide range of engineering disciplines. The most frequently used orthogonal wavelets are Haar, Legendre, Chebyshev, the second kind of Chebyshev and CAS wavelets. We use this orthogonal basis for the purpose of reducing the problem under consideration to a system of linear or nonlinear algebraic equations. The Large systems of algebraic equations may result in greater computational complexity and large storage requirements. However the operational matrix for the Legendre wavelets is structurally

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spare. This reduces the computational complexity of the resulting algebraic system. Legendre wavelets method as a specific kind of wavelets methods has been widely applied for solving differential equations [25–29].

The main purpose of this paper is to solve the following general form of the system of FDEs [13] by Legendre wavelets:

$$D_*^{\alpha_i} u_i(x) = f_i(x, u_1, u_2, \ldots, u_n), \quad u_i^{(r)}(0) = c_i, \quad 1 \leqslant i \leqslant n, \quad 0 \leqslant r \leqslant \lceil \alpha_i \rceil, \tag{1}$$

where $m_i - 1 < \alpha_i \leqslant m_i,\ m_i \in \mathbb{Z}^+$, $D_*^{\alpha_i}$ is the Caputo fractional differential operator. The existence, uniqueness and stability of solutions of FDEs.(1) with initial values are proved in [12,13]. Many authors solved this system by using different numerical methods [21–24]. While there are no articles about solving the system as above using wavelets method. So in this paper, we take the method of Legendre wavelets to the numerical solutions of FDEs.(1) into consideration.

This article is organized as follows. In Section 2, we introduce the definitions of fractional calculus simply. In Section 3, we describe the basic formulation of Legendre wavelets and their properties, and obtained the operational matrix of Legendre wavelets through the operational of Block Pulse Functions (BPFs). Section 4, present the convergence and error analysis of Legendre wavelets method. Section 5, Legendre wavelets method is used to the numerical solutions of three examples of FDEs (1). Finally a conclusion is given in Section 6.

2. Preliminaries and notions

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration [29]. There are many different types of definitions of fractional calculus. For example, the Riemann–Liouville integral operator of order α is defined by [29]:

$$(I^\alpha u)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, & \alpha > 0,\ \tau > 0, \\ u(t), & \alpha = 0, \end{cases} \tag{2}$$

and its fractional derivative of order α ($\alpha \geqslant 0$) is normally used:

$$(D_*^\alpha u)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha} u)(t), \quad \alpha > 0,\ n-1 < \alpha \leqslant n. \tag{3}$$

In this article we adopt the Caputo's definition, which is a modification of Riemann–Liouville definition:

$$(D_*^\alpha u)(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau, & \alpha > 0, n-1 < \alpha < n, \\ \frac{d^n u(t)}{dt^n}, & \alpha = n, \end{cases} \tag{4}$$

where n is an integer. Caputo's integral operator has useful properties:

$$(D_*^\alpha I^\alpha u)(t) = u(t), \tag{5}$$

$$(I^\alpha D_*^\alpha u)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0^+)}{k!} t^k, \quad t \geqslant 0,\ n-1 < \alpha < n. \tag{6}$$

In this study, the fractional derivative is understood in the Caputo sense because of its applicability to real-world problems.

3. The Legendre wavelets and their properties

3.1. Legendre polynomial and Legendre wavelets

The well-known Legendre polynomials $L_n(x)$ are orthogonal with respect to the weight function $\omega(x) = 1$ on the interval $[-1, 1]$ and satisfy the following recurrence formulae [25]:

$$L_0 = 1, \quad L_1 = x, \quad L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x), \quad n = 1, 2, \ldots, \tag{7}$$

For practical use of polynomials on the interval [0, 1], it is necessary to shift the defining domain by means of the substitution $x = 2t - 1$ ($0 \leqslant t \leqslant 1$), so the shifted Legendre polynomial $L_n^*(t)$ defined on [0, 1] as $L_n^*(t) = L_n(2t-1)$.

It also satisfy the orthogonality condition as

$$\int_0^1 L_n^*(t) L_m^*(t) dt = \frac{1}{2n+1} \delta_{nm}, \tag{8}$$

where δ_{nm} is the Kronecker delta.

Legendre wavelets $\psi_{nm}(t) = \psi(k, \tilde{n}, m, t)$ have four arguments, k can be assumed as any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval [0, 1) as: