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Constraint Optimal Selection Techniques (COSTs) for nonnegative linear programming problems

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ABSTRACT

We describe an active-set, cutting-plane approach called Constraint Optimal Selection Techniques (COSTs) and develop an efficient new COST for solving nonnegative linear programming problems. We give a geometric interpretation of the new selection rule and provide computational comparisons of the new COST with existing linear programming algorithms for some large-scale sample problems.

1. Introduction

1.1. The nonnegative linear programming problem

Linear programming represents a mathematical model for solving numerous practical problems such as the optimal allocation of resources. The general linear programming (LP) model [1] can be stated as the problem P

$$\text{maximize } z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{1}$$

$$\text{subject to } \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \tag{2}$$

$$\mathbf{x} \geqslant \mathbf{0}, \tag{3}$$

where $\mathbf{x}$ is an n-dimensional column vector of variables; $\mathbf{A}$ is an $m \times n$ matrix $[a_{ij}]$ with m rows of transposed n-dimensional column vectors $\mathbf{a}_i^{\mathrm{T}},\ \forall i = 1, \ldots, m;$ $\mathbf{b}$ is an m-dimensional column vector; $\mathbf{c}$ is an n-dimensional column vector; and $\mathbf{0}$ is a column vector of zeros with its dimension apparent from context. Simplex pivoting algorithms and polynomial interior-point barrier-function methods represent the two principal solution approaches to solve problem P. Unfortunately there is no single best algorithm. For either method, we can always formulate an instance of P for which the method performs poorly [2]. Significantly, however, binary and integer models represent the principal use of LP in industrial applications. Since interior-point methods do not allow the efficient post-optimality analysis of pivoting algorithms in solving such problems, simplex methods remain the dominant approach. Yet current simplex algorithms are not adequate in many situations. In particular, emerging technologies require computer solutions in nearly real time for problems involving millions of constraints or variables.

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In this paper we consider the special case of P with $\mathbf{a}_i \geqslant \mathbf{0}$ and $\mathbf{a}_i \neq \mathbf{0},\ \forall i = 1,\ldots,m;\ \mathbf{b} > \mathbf{0}$; and $\mathbf{c} > \mathbf{0}$, where all inequalities between vectors are meant componentwise. Such a version of P is called a nonnegative linear program (NNLP) and provides a model for a significant portion of industrial LP applications. Examples include updating airline schedules as weather conditions and passenger loads change [3], finding an optimal driving route using real-time traffic data from global positioning satellites [4], and detecting problematic repeats in DNA sequences [5]. The dual of the NNLP maximization problem (1), (2), and (3) is considered the standard minimization NNLP problem. We focus here on the maximization case.

NNLPs have the following two useful properties. They are always feasible at the origin $\mathbf{x} = \mathbf{0}$. Moreover, for each $j = 1,\ldots,n$,

$$x_j \leqslant \min_{i=1,\ldots,m}\left\{\frac{b_i}{a_{ij}}\middle| a_{ij} > 0\right\}.$$

Hence NNLPs have both an unbounded feasible region and unbounded objective function if and only if some column of $\mathbf{A}$ is a zero vector. Thus their boundedness is easily verifiable without computation.

1.2. An active-set framework

Our active-set framework for solving an NNLP P is described as follows. For P with a bounded feasible region as determined above, we begin with a relaxation of P. We next form an initial bounded NNLP P_0 by one of two methods. In the single-bound approach P_0 has a single artificial bounding constraint $\mathbf{a}_0^T\mathbf{x} \leqslant b_0$ in (2) with $\mathbf{a}_0 > \mathbf{0}$ and $b_0 > 0$, together with the nonnegativity constraints (3). One well-known bounding constraint has the form $\mathbf{1x} \leqslant M$ for M sufficiently large enough so as not to reduce the feasible region of P. In the multi-bound approach described in Section 2.2, the initial bounding constraints are selected in the same manner as later constraints. Computational results, both reported and otherwise, show little difference in the CPU times for solving problems using these two approaches except when the solution to P_0 with the above artificial bounding constraint produces a large number of alternate optima. Here the multi-bound approach is used for P_0 unless otherwise stated since it also considers both the bounding and multi-cut constraint selection aspects of our method.

We solve P_0 and a series of relaxations $P_r,\ r = 1,2,\ldots$, of P by adding constraints from set (2) to the previous relaxation. The constraints of P_r are called its *operative constraints*, while the rest of the constraints (2) are called its *inoperative constraints*. Since the bounding constraints and nonnegativity restrictions form a bounded region, each problem P_r yields an optimal solution $\mathbf{x}_r^*$. A relaxed problem P_{r+1} is obtained from P_r by adding one or more constraints chosen from the violated inoperative constraints of P_r, i.e., a constraint $\mathbf{a}_i^T\mathbf{x} \leqslant b_i$ not in P_r for which $\mathbf{a}_i^T\mathbf{x}_r^* - b_i > 0$. These constraints are selected according to a particular criterion such that the chosen violated inoperative constraints are considered most likely to be binding at optimality for the original problem P. P_{r+1} is then solved by the dual simplex algorithm. By continuing in this manner, eventually a solution $\mathbf{x}_r^*$ is obtained that satisfies all inoperative constraints for P_r and yields a solution of P. The rationale for any such active-set approach is that a solution to P is determined by relatively few constraints satisfied as equalities — at most n such constraints for the n variables in constraint set (2). Therefore, the goal is to add only constraints likely to be binding at optimality.

1.3. Background

Active-set approaches for solving P have been studied by Stone [6], Thompson et al. [7], Adler et al. [8], Zeleny [9], Myers and Shih [10], Curet [11], and Bixby et al. [12], with the term "constraint selection technique" used in Myers and Shih [10]. In such work, however, the only two selection criteria used for selecting an inoperative constraint violated by the solution to the current relaxed problem were (a) to randomly select a constraint from the violated inoperative constraint and (b) to choose an inoperative constraint most violated by the current solution. For example, method (a) was used in Adler et al. [8], while method (b) was used in Zeleny [9] and Mitchell [13]. If P_0 has a bounding constraint $\mathbf{c}^T\mathbf{x} \leqslant M$ and the remaining constraints of (2) are randomly ordered, then method (a) is called SUB in this paper because it selects the violated inoperative constraint with the smallest SUBscript in (2). Similarly, method (b) is called VIOL because it selects the most VIOLated inoperative constraint. In this paper we emphasize the comparison with SUB and VIOL since they are the active-set approaches prevalent in the literature. In particular, VIOL is identical to the Priority Constraint Method of Thompson et al. [7] and is a standard pricing method for adding constraints in cutting plane methods [13] and for adding variables in delayed column generation in terms of the dual [1,12]. Bixby et al. [12] develops a pricing rule, called *sifting*, based upon a scaled violation of the constraints in the dual problem. As a subroutine within a predictor–corrector interior point cutting plane algorithm, Mitchell [13] uses a multi-cut version of the VIOL criteria to select the cutting planes.

Recent work on constraint selection has considered the angle that the normal vector $\mathbf{a}_i$ of a constraint $\mathbf{a}_i^T\mathbf{x} \leqslant b_i$ in (2) forms with the normal vector $\mathbf{c}$ of the objective function (1) as measured by the cosine of the angle between $\mathbf{a}_i$ and $\mathbf{c}$ denoted by $\cos(\mathbf{a}_i, \mathbf{c}) = \frac{\mathbf{a}_i^T\mathbf{c}}{\|\mathbf{a}_i\|\|\mathbf{c}\|}$. Geometric considerations in Naylor and Sell [14], for example, suggest that a constraint $\mathbf{a}_i^T\mathbf{x} \leqslant b_i$ with a larger cosine value might more likely be binding at an optimal extreme point of P than one with a smaller value. We call this observation the cosine criterion, which applied to the dual of P gives new pivot rules for the simplex algorithm such as the "most-obtuse-angle" rules studied by Pan [15,16]. In addition, Trigos et al. [17] and Vieira et al. [18] use the cosine criterion on P for both (2) and (3) to get an initial basis for the simplex algorithm and a reduction in the number of iterations. Corley