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A numerical method for solving elasticity equations with interface involving multi-domains and triple junction points

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ABSTRACT

Solving elasticity equations with interfaces on multiple domains has wide applications in engineering and science. Corner singularities make it difficult for most existing solvers to deal with a triple junction in the case of nonelastic problems. Therefore constructing an efficient and accurate solver for an elasticity problem with multiple domains is a challenge. In this paper, an efficient non-traditional finite element method with non-body-fitted grids is proposed to solve elliptical elasticity equations with multi-domains and triple junction points. Numerical experiments show that this method is approximately second order accurate in the L^∞ norm for piecewise smooth solutions.

1. Introduction

Problems involving elliptical elasticity equations with interfaces have a wide variety of applications in science and engineering. Designing highly effective and computational efficient methods for these problems is nontrivial, especially those involving multi-domains and triple junctions. Having an effective solver for that type of problems would have major applications to material science problems, for example Hele–Shaw flow for microfluidics [7].

Much work has been done on mass transport through fluid interfaces for biomedical applications [8]. Biomolecule surfaces can serve as interfaces, for example the Poisson–Boltzman model for electrostatic analysis, Nerst–Planck equation for conservation of mass of ions in a fluid medium, and the combination of the two, Poisson–Nernst–Plank model for charge permeation.

These applications are derived from looking at simple diffusion laws involving an interface. Let J be the diffusion flux, D the diffusivity, and ϕ the concentration.

Fick's First Law of Diffusion states that $J = -D\frac{\partial\phi}{\partial x}$.

Fick's Second Law of Diffusion states $\frac{\partial\phi(x,t)}{\partial t} = \nabla\cdot(D(x)\nabla\phi(x,t))$.

For example, we could explore the steady state solution to this time dependent partial differential equation.

Consider an open bounded domain $\Omega\subset\mathbb{R}^d$. Let Γ be an interface of codimension $d-1$, which divides Ω into disjoint open subdomains Ω_1, Ω_2, and Ω_3, hence $\Omega = \Omega_1\cup\Omega_2\cup\Omega_3$, see Fig. 1. Assume that the boundary, $\partial\Omega$ and the boundary of each subdomain $\partial\Omega_{1,2,3}$ are Lipschitz continuous. Since $\partial\Omega_{1,2,3}$ is Lipschitz, so is Γ. A unit normal vector n can be defined on Γ a.e. on Γ, see Section 1.5 on [3]. $x = (x_1, x_2, \ldots, x_d)$ and ∇ is the gradient operator. The coefficient, β is assumed to be a d by d matrix, that is uniformly elliptic on each disjoint subdomain of Ω.

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We seek solutions to the variable coefficient elliptical equations away from the interface of the form given by:
For $x \in \Omega_i,\ i = 1, 2, 3,$

$$\begin{aligned} &-\nabla\cdot(\beta_1^i\nabla u_1^i) - \nabla\cdot(\beta_2^i\nabla u_2^i) = f_1^i, \\ &-\nabla\cdot(\beta_3^i\nabla u_1^i) - \nabla\cdot(\beta_4^i\nabla u_2^i) = f_2^i. \end{aligned} \tag{1}$$

Boundary conditions:

$$u_i = g_i \quad \text{on } \partial\Omega\cup\Omega_i. \tag{2}$$

And the jump conditions are given by:
For $x \in \Gamma_1$,

$$\begin{aligned} a_1^1(x) &= u_1^2 - u_1^3, \\ a_2^1(x) &= u_2^2 - u_2^3, \\ b_1^1(x) &= n_1\cdot(\beta_1^2(x)\nabla u_1^2 + \beta_2^2(x)\nabla u_2^2) - n_1\cdot(\beta_1^3(x)\nabla u_1^3 + \beta_2^3(x)\nabla u_2^3), \\ b_2^1(x) &= n_1\cdot(\beta_3^2(x)\nabla u_1^2 + \beta_4^2(x)\nabla u_2^2) - n_1\cdot(\beta_3^3(x)\nabla u_1^3 + \beta_4^3(x)\nabla u_2^3). \end{aligned} \tag{3}$$

For $x \in \Gamma_2$,

$$\begin{aligned} a_1^2(x) &= u_1^3 - u_1^1, \\ a_2^2(x) &= u_2^3 - u_2^1, \\ b_1^2(x) &= n_2\cdot(\beta_1^3(x)\nabla u_1^3 + \beta_2^3(x)\nabla u_2^3) - n_2\cdot(\beta_1^1(x)\nabla u_1^1 + \beta_2^1(x)\nabla u_2^1), \\ b_2^2(x) &= n_2\cdot(\beta_3^3(x)\nabla u_1^3 + \beta_4^3(x)\nabla u_2^3) - n_2\cdot(\beta_3^1(x)\nabla u_1^1 + \beta_4^1(x)\nabla u_2^1). \end{aligned} \tag{4}$$

For $x \in \Gamma_3$,

$$\begin{aligned} a_1^3(x) &= u_1^1 - u_1^2, \\ a_2^3(x) &= u_2^1 - u_2^2, \\ b_1^3(x) &= n_3\cdot(\beta_1^1(x)\nabla u_1^1 + \beta_2^1(x)\nabla u_2^1) - n_3\cdot(\beta_1^2(x)\nabla u_1^2 + \beta_2^2(x)\nabla u_2^2), \\ b_2^3(x) &= n_3\cdot(\beta_3^1(x)\nabla u_1^1 + \beta_4^1(x)\nabla u_2^1) - n_3\cdot(\beta_3^2(x)\nabla u_1^2 + \beta_4^2(x)\nabla u_2^2). \end{aligned} \tag{5}$$

For nearly four decades, research has been performed in the area of numerical solutions of elliptical equations with discontinuous coefficients and singular sources on Cartesian grids. It started with the pioneering work of Peskin [14] on the first order accurate immersed boundary method, which he developed to simulate the pattern of blood flow to the heart.

Over that large time span, a great amount of work has been done to use finite difference methods on elliptical interface problems. The main idea is to use difference schemes and stencils near the interface to incorporate the jump conditions and interface in the Taylor expansions. Using finite difference schemes requires the use of high order derivatives of jump conditions and interface conditions.

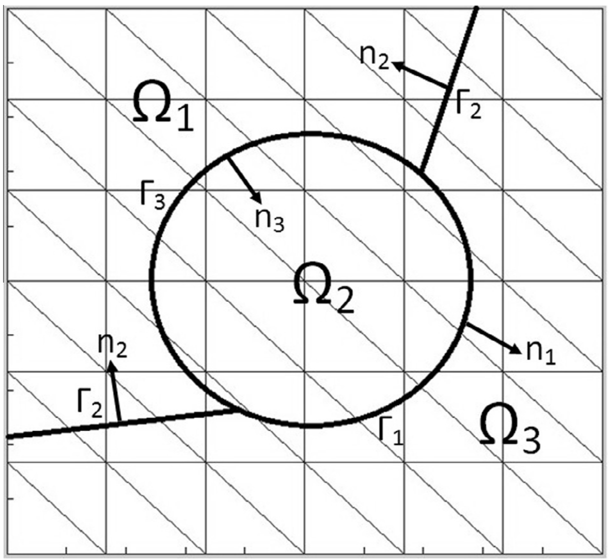

Fig. 1. A uniform triangulation.