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On a subclass of the class of rapidly varying sequences

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ABSTRACT

We define and study the class $\mathsf{KR}_{\infty,s}$ which is a proper subclass of the class $\mathsf{R}_{\infty,s}$ of rapidly varying sequences of index of variability ∞. Then, we prove a theorem of the Galambos–Bojanić–Seneta type for this class, as well as a representation theorem in the Karamata sense. We also give some important asymptotic properties and characterizations of sequences in $\mathsf{KR}_{\infty,s}$.

1. Introduction

Karamata theory of regular variability (and various its generalizations and variations) is a very important part of mathematical analysis, especially of asymptotic analysis [1]. The main object of this theory is the notion of slowly varying function.

A function $g : [a,\infty) \to (0,\infty), a > 0$, is said to be *slowly varying* in the sense of Karamata [15] if it is measurable and for each $\lambda > 0$ satisfies

$$\lim_{x\to\infty} \frac{g(\lambda x)}{g(x)} = 1. \tag{1}$$

We denote the class of slowly varying functions by SV_f.

Another theory, conjugate with the theory of slow variability, is de Haan's theory of rapid variability. (These theories are conjugated, for example, through generalized inverse [11].)

A function $g : [a,\infty) \to (0,\infty), a > 0$, is said to be *rapidly varying* of index of variability ∞ [14] if it is measurable and for each $\lambda > 1$ satisfies

$$\lim_{x\to\infty} \frac{g(\lambda x)}{g(x)} = \infty. \tag{2}$$

We denote the class of rapidly varying functions by $\mathsf{R}_{\infty,f}$.

Both theories, the theory of regular and rapid variability, have a sequential analog (see [1,3–5,7–9,20] in connection with these two theories).

A sequence $(c_n)_{n\in\mathbb{N}}$ of positive real numbers is said to be *rapidly varying* (in the sense of de Haan) of index of variability ∞ if for each $\lambda > 1$ it satisfies

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$$\lim_{n\to\infty}\frac{c_{[\lambda n]}}{c_n}=\infty. \quad (3)$$

We denote the class of rapidly varying sequences by $\mathsf{R}_{\infty,s}$ (see [3]).

In this paper we study an important subclass of the class $\mathsf{R}_{\infty,s}$, that we denote by $\mathsf{KR}_{\infty,s}$.

For a sequence $\mathbf{c}=(c_n)_{n\in\mathbb{N}}$ of positive real numbers the *lower Matuszewska index* $d(\mathbf{c})$ is defined as the supremum of all $d\in\mathbb{R}$ such that for each $\Lambda>1$

$$\frac{c_{[\lambda n]}}{c_n}\geqslant \lambda^d(1+o(1)) \quad (n\to\infty), \quad (4)$$

holds uniformly (with respect to λ) on the segment $[1,\Lambda]$ (compare with the definition of lower Matuszewska index for functions [1, p. 68]). The sequence $\mathbf{c}$ belongs to the *class* $\mathsf{KR}_{\infty,s}$ if $d(\mathbf{c})=\infty$.

Let us mention that in a similar way one defines the *lower Matuszewska index* $d(g)$ of a measurable function $g:[a,\infty)\to(0,\infty)$. The class of all measurable functions whose lower Matuszewska index is ∞ is denoted by $\mathsf{KR}_{\infty f}$. This class of functions has very important asymptotic properties (see [1] in this connection). By a result from [1] we have $\mathsf{KR}_{\infty f}\subsetneqq\mathsf{R}_{\infty f}$.

The theory of regular and rapid variability has many applications in different branches of mathematics: differential and difference equations, in particular in description of asymptotic properties of solutions of these equations, time scales theory, dynamic equations, q-calculus, probability theory, number theory and so on (see, for instance, [17–19,21]).

It is natural to expect that the class $\mathsf{KR}_{\infty,s}$ may also have many applications. This hope is based, among other facts, on the Seneta-de Haan theorem [1, Theorem 2.4.7] which gives nice relations between classes SV_f and $\mathsf{KR}_{\infty f}$ under the generalized inverse. Also, there is the connection between rapidly varying functions and their cumulative maximum functions from the class $\mathsf{KR}_{\infty f}$ (see [1, p. 87], in particular Proposition 2.4.6).

2. Results

Theorem 2.1. *For a sequence* $\mathbf{c}=(c_n)_{n\in\mathbb{N}}$ *of positive real numbers the following are equivalent:*

(1) $c\in\mathsf{KR}_{\infty,s}$;
(2) *For each* $d\in\mathbb{R}$ *it holds* $\liminf_{n\to\infty}\inf_{\lambda\geqslant1}\frac{c_{[\lambda n]}}{\lambda^d c_n}\geqslant1$.

Proof. $(1)\Rightarrow(2)$ From $d(\mathbf{c})=\infty$, it follows that for every $d\in\mathbb{R}$, every $\Lambda>1$, and sufficiently large n we have $\frac{c_{[\lambda n]}}{c_n}\geqslant\lambda^d(1+o(1))$, where $\lambda\in[1,\Lambda]$ is an arbitrary fixed element. For the same d,λ,Λ, for sufficiently large n we have $\inf_{\lambda\in[1,\Lambda]}\frac{c_{[\lambda n]}}{\lambda^d c_n}\geqslant1+o(1)$. In other words, for each $\varepsilon>0$ there is $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $\inf_{\lambda\in[1,\Lambda]}\frac{c_{[\lambda n]}}{\lambda^d c_n}\geqslant1-\varepsilon$ for each $n\geqslant n_0$. Because the last inequality is true for each $\Lambda>1$, it follows that (for the same d) for each $\lambda\geqslant1$ we have $\inf_{\lambda\geqslant1}\frac{c_{[\lambda n]}}{\lambda^d c_n}\geqslant1-\varepsilon$. As ε was arbitrary (2) follows.

$(2)\Rightarrow(1)$ Suppose that for an arbitrarily fixed $d\in\mathbb{R}$, $\liminf_{n\to\infty}\inf_{\lambda\geqslant1}\frac{c_{[\lambda n]}}{\lambda^d c_n}\geqslant1$ is satisfied. Then for the same d and each $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $\inf_{\lambda\geqslant1}\frac{c_{[\lambda n]}}{\lambda^d c_n}\geqslant1-\varepsilon$ for each $n\geqslant n_0$. In other words, for the same d,ε,n_0, and for each $\lambda\geqslant1$, especially for $\lambda\in[1,\Lambda]$, $\Lambda>1$ an arbitrary real number, it holds $\frac{c_{[\lambda n]}}{c_n}\geqslant\lambda^d(1-\varepsilon)$ for each $n\geqslant n_0$. This means that for each $\Lambda>1$ we have $\frac{c_{[\lambda n]}}{c_n}\geqslant\lambda^d(1+o(1))$ uniformly with respect to $\lambda\in[1,\Lambda]$ for $n\to\infty$. Since d is arbitrary, (1) follows. □

The next statement is a result of the Galambos–Bojanić–Seneta type (see [1,2,6,10,12,13]).

Theorem 2.2. *For a sequence* $\mathbf{c}=(c_n)_{n\in\mathbb{N}}$ *of positive real numbers the following are equivalent:*

(1) $\mathbf{c}\in\mathsf{KR}_{\infty,s}$;
(2) *The function* $g(x)=c_{[x]}, x\geqslant1$, *belongs to the class* $\mathsf{KR}_{\infty f}$.

Proof. $(1)\Rightarrow(2)$ Let $\mathbf{c}=(c_n)_{n\in\mathbb{N}}\in\mathsf{KR}_{\infty,s}$. Then by Theorem 2.1 we have $\liminf_{n\to\infty}\inf_{\lambda\geqslant1}\frac{c_{[\lambda n]}}{\lambda^d c_n}\geqslant1$ for each $d\in\mathbb{R}$. This means that for the same d and each $\varepsilon>0$ there is $n_0=n_0(d,\varepsilon)\in\mathbb{N}$ such that $\inf_{\lambda\geqslant1}\frac{c_{[\lambda[x]]}}{\lambda^d c_{[x]}}\geqslant1-\varepsilon$ for each $x\geqslant n_0$ $(\geqslant1)$. Therefore, for the same d,ε,n_0 it is true

$$\inf_{\lambda\geqslant1}\frac{c_{[\lambda x]}}{\lambda^d c_{[x]}}=\inf_{\lambda\geqslant1}\frac{c_{\left[\frac{x}{[x]}\cdot[x]\cdot\lambda\right]}}{\lambda^d c_{[x]}}\geqslant\inf_{\lambda\geqslant1}\frac{c_{[[x]\cdot\lambda]}}{\lambda^d c_{[x]}}\geqslant1-\varepsilon,$$

i.e. (for this d)

$$\liminf_{x\to\infty}\inf_{\lambda\geqslant1}\frac{c_{[\lambda x]}}{\lambda^d c_{[x]}}\geqslant1.$$