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A deformed reduced semi-discrete Kaup–Newell equation, the related integrable family and Darboux transformation

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ABSTRACT

A deformed reduced semi-discrete Kaup–Newell equation and its related integrable family are derived from discrete zero curvature equation. A Darboux transformation of Lax pair of this equation is established with the help of gauge transformation. By means of the resulting Darboux transformation, three exact solutions are given.

1. Introduction

Beginning from the original work of Fermi, Past and Ulam in the 1960s [1], the integrable differential–difference equations (or the lattice soliton equations) have received considerable attention. Many integrable differential–difference equations have been presented [2–6]. Much investigation on the integrable differential–difference equations has been obtained [2–23], such as the inverse scattering transformation [2], the symmetries and master symmetries [7–9], Hamiltonian structures [10–16], integrable coupling systems [13–16], nonlinearization of the Lax pairs [17,18], constructing complexiton solutions by the Casorati determinant [19], the Darboux transformations [20–22], and so on. For a function $f_n = f(n,t)$,the shift operator E, the inverse of E and the difference operator D are defined by

$$Ef_n = f(n+1,t), \quad E^{-1}f_n = f(n-1,t), \quad Df_n = f(n+1,t) - f(n,t), \quad n \in Z. \tag{1}$$

In Ref. [23], a reduced semi-discrete Kaup–Newell equation

$$\begin{cases} r_{n_t} = \frac{r_n}{1+r_n s_{n+1}} - \frac{r_{n-1}}{1+r_{n-1} s_n}, \\ s_{n_t} = \frac{s_{n+1}}{1+r_n s_{n+1}} - \frac{s_n}{1+r_{n-1} s_n} \end{cases} \tag{2}$$

is introduced, its r-Matrix and conserved quantities are presented. In Eq. (2), if we use r_{n+1} instead of r_n, and apply E^{-1} in the first equation, Eq. (2) becomes

$$\begin{cases} r_{n_t} = \frac{r_n}{1+r_n s_n} - \frac{r_{n-1}}{1+r_{n-1} s_{n-1}}, \\ s_{n_t} = \frac{s_{n+1}}{1+r_{n+1} s_{n+1}} - \frac{s_n}{1+r_n s_n}. \end{cases} \tag{3}$$

Therefore, Eq. (3) is a deformed reduced semi-discrete Kaup–Newell equation. In this letter we would like to research Darboux transformation of Lax pair of the deformed reduced semi-discrete Kaup–Newell equation (3). As is well known,Darboux transformation of Lax pair is an important method to find exact solutions of the integrable differential–difference equations. Unlike some complicated analytical methods, Darboux transformation is a powerful pure algebraic method. Furthermore, a

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remarkable feature of many integrable differential–difference equations is that they are members of integrable families, every family makes up of an infinite sequence of integrable differential–difference equations. The members in the same family share identical spatial part of Lax pair and can be solved by the same procedure via inverse scattering method. Therefore, it is important to find out which family the deformed reduced semi-discrete Kaup–Newell equation (3) belongs to. In what follows, we not only derive the Lax pair of Eq. (3), but also deduce the corresponding integrable family of Eq. (3).

This paper is organized as follows. In Section 2, we introduce a matrix spectral problem

$$E\varphi_n = U_n(u_n,\lambda)\varphi_n, U_n(u_n,\lambda) = \begin{pmatrix} 1 & r_n\lambda \\ s_n\lambda & \lambda^2+1+r_ns_n\lambda^2 \end{pmatrix}, \tag{4}$$

$\varphi_n = (\varphi_n^1, \varphi_n^2)^T$ is eigenfunction vector, λ is the spectral parameter and $\lambda_t = 0$, $(r_n, s_n)^T$ is the potential vector, and $r_n = r(n,t)$, $s_n = s(n,t)$ depend on integer $n \in Z$ and real $t \in R$. Make use of the discrete zero curvature representation, we are going to derive a family of integrable differential–difference equations. In obtained family, the typical member is the deformed reduced semi-discrete Kaup–Newell equation (3). In Section 3, a Darboux transformation is constructed by means of the gauge transformation of Lax pairs for the deformed reduced semi-discrete Kaup–Newell equation (3). In Section 4, as applications of Darboux transformation, three exact solutions of Eq. (3) are given. Finally, in Section 5, there will be some conclusions and remarks.

2. The family of integrable differential–difference equations

In this section, we shall derive a family of integrable differential–difference equations associated with eigenvalue problem (4). To this end, we first solve the following stationary discrete zero curvature equation

$$(E\chi_n)U_n - U_n\chi_n = \chi_{n+1}U_n - U_n\chi_n = 0. \tag{5}$$

Upon setting

$$\chi_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix}.$$

We find that Eq. (5)becomes

$$\begin{aligned} &(A_{n+1}-A_n) + (-r_nC_n + s_nB_{n+1})\lambda = 0, \\ &(1+r_ns_n)B_{n+1}\lambda^2 + r_n(A_{n+1}+A_n)\lambda - B_n + B_{n+1} = 0, \\ &(1+r_ns_n)C_n\lambda^2 + s_n(A_{n+1}+A_n)\lambda - C_{n+1} + C_n = 0, \\ &(1+r_ns_n)(A_n - A_{n+1})\lambda^2 + (r_nC_{n+1} - s_nB_n)\lambda + (A_n - A_{n+1}) = 0. \end{aligned} \tag{6}$$

Substituting expansions

$$A_n = \sum_{m=0}^{\infty} A_n^{(m)}\lambda^{-2m}, \quad B_n = \sum_{m=0}^{\infty} B_n^{(m)}\lambda^{-2m+1}, \quad C_n = \sum_{m=0}^{\infty} C_n^{(m)}\lambda^{-2m+1}$$

into Eq. (6) and comparing the coefficients of λ^i, $i = 0, 1, 2, \ldots$, in Eq. (6), we obtain the initial conditions:

$$(1+r_ns_n)(A_{n+1}^{(0)} - A_n^{(0)}) = r_nC_{n+1}^{(0)} - s_nB_n^{(0)}, \quad (1+r_ns_n)B_{n+1}^{(0)} = 0, \quad (1+r_ns_n)C_n^{(0)} = 0$$

and the recursion relations:

$$\begin{aligned} &(1+r_ns_n)\left(A_{n+1}^{(m+1)} - A_n^{(m+1)}\right) = -s_nB_n^{(m+1)} + r_nC_{n+1}^{(m+1)} + \left(A_n^{(m)} - A_{n+1}^{(m)}\right) \quad m \geqslant 0, \\ &(1+r_ns_n)B_{n+1}^{(m+1)} = -r_n(A_n^{(m)} + A_{n+1}^{(m)}) + B_n^{(m)} - B_{n+1}^{(m)}, \quad m \geqslant 0, \\ &(1+r_ns_n)C_n^{(m+1)} = -s_n(A_n^{(m)} + A_{n+1}^{(m)}) + C_{n+1}^{(m)} - C_n^{(m)}, \quad m \geqslant 0. \end{aligned} \tag{7}$$

Proposition 1. *If the initial values are chosen as follows*

$$A_n^{(0)} = -1/2, \quad B_n^{(0)} = 0,$$

then $A_n^{(m)}$, $B_n^{(m)}$, $C_n^{(m)}$, $m \geqslant 0$, *which are solved by Eq. (7), are all local, and they are just rational functions in the two dependent variables* r_n *and* s_n.

Proof. On the basis of second and third equations in Eq. (7), we see that $B_n^{(m+1)}$ and $C_n^{(m+1)}$ can be determined locally by $A_n^{(m)}$, $B_n^{(m)}$ and $C_n^{(m)}$, $m \geqslant 0$. In order to obtain $A_n^{(m+1)}$, $m \geqslant 0$ from first equation in Eq. (7), we need to use operator $D^{-1} = (E-1)^{-1}$ to solve the corresponding difference equation. In what follows, we will show that $A_n^{(m+1)}$, $m \geqslant 0$ may be deduced through an algebraic method rather than by solving the difference equation. From Eq. (5), we know that