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Existence of positive periodic solutions to nonlinear integro-differential equations

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ABSTRACT

This paper deals with the existence of positive periodic solutions for a class of nonlinear integro-differential equations. Such equations arise in the theory of a circulating fuel nuclear reactor. The existence of positive solutions and exponential stability is also treated. The main results are illustrated with several examples.

1. Introduction

In this paper we investigate the existence of positive ω-periodic solutions for the nonlinear integro-differential equations of the form

$$\dot{x}(t) + \int_{t-\tau}^{t} p(t-s)g(x(s))\,ds = 0, \quad t \geqslant T. \tag{1}$$

With respect to Eq. (1) in the next sections we will assume the following conditions:

(i) $p \in C([0,\tau), R)$,
(ii) $g \in C((0,\infty),(0,\infty))$.

The Eq. (1) was encountered by [1] in the theory of a circulating fuel nuclear reactor. In this model, x is the neutron density. It is also a good model in one dimensional viscoelasticity in which x is the strain and p is the relaxation function. Such equations have been also proposed to model some biological problems.

The stability of zero solution of Eq. (1) was later studied in [2–6] and the references cited therein. To the best of our knowledge there are only a few results on the existence of periodic solutions of Eq. (1). According to applications it is reasonable to consider the positive solutions of (1). In this paper we will obtain existence criteria for the positive ω-periodic solutions of Eq. (1). For such equations the existence results in the literature are largely based on the several assumptions for the function $p(t)$ and the authors usually do not consider the existence of positive periodic solutions. For example the authors in [3,5] assume that the function $p(t)$ satisfies the next conditions: $p(\tau) = 0$, $p(t) \geqslant 0$, $\dot{p}(t) \leqslant 0$, $\ddot{p}(t) \equiv 0$, $0 \leqslant t \leqslant \tau$. Authors in [7] studied oscillatory solutions of linear integro-differential equations. Motivated by the discussion above, we will focus on the existence of positive periodic solutions for the Eq. (1) without assumptions above on the function $p(t)$. For related results see also [8,9] and the references therein. In the third section the existence of positive solutions is also considered. In Section 4 we will establish the conditions for the exponential stability of positive solution of Eq. (1). To the

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best of our knowledge this problem has not been solved yet. For related results we refer readers to [10,11] and the references cited therein.

The following fixed point theorem will be used to prove the main results in the next sections.

Theorem 1.1 (*Schauder's Fixed Point Theorem [12,13]*). *Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S:\Omega\to\Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x\in\Omega$ such that $Sx=x$.*

2. Existence of periodic solutions

In this section we will study the existence of positive ω-periodic solutions of Eq. (1). In the next lemma and theorems we choose $T>\tau>0$.

Lemma 2.1. *Suppose that there exists a positive and continuous function $k(t,s),\ t-\tau\leqslant s\leqslant t$, such that*

$$\int_t^{t+\omega}\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du=0,\quad t\geqslant T. \tag{2}$$

Then the function

$$f(t)=\exp\left(-\int_T^t\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right),\quad t\geqslant T,$$

is ω-periodic.

Proof. For $t\geqslant T$ we obtain

$$\begin{aligned}f(t+\omega)&=\exp\left(-\int_T^{t+\omega}\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right)\\&=\exp\left(-\int_T^{t}\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right)\times\exp\left(-\int_t^{t+\omega}\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right)=f(t).\end{aligned}$$

Thus the function $f(t)$ is ω-periodic.

Theorem 2.1. *Suppose that there exists a positive and continuous function $k(t,s),\ t-\tau\leqslant s\leqslant t$, such that (2) holds and*

$$\exp\left(\int_T^t\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right)\times g\left(\exp\left(-\int_T^s\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right)\right)=k(t,s),\quad t\geqslant T. \tag{3}$$

Then Eq. (1) has a positive ω-periodic solution.

Proof. Let $X=\{x\in C([T-\tau,\infty),R)\}$ be the Banach space with the norm $\|x\|=\sup_{t\geqslant T-\tau}|x(t)|$. We set

$$f(t)=\exp\left(-\int_T^t\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right),\quad t\geqslant T.$$

With regard to Lemma 2.1 we have $m\leqslant f(t)\leqslant M$, where

$$\begin{aligned}m&=\min_{t\in[T,\infty)}\left\{\exp\left(-\int_T^t\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right)\right\},\\M&=\max_{t\in[T,\infty)}\left\{\exp\left(-\int_T^t\int_{u-\tau}^{u} p(u-v)k(u,v)\,dv\,du\right)\right\}.\end{aligned} \tag{4}$$

We now define a closed, bounded and convex subset Ω of X as follows

$$\begin{aligned}&\Omega=\{x\in X: x(t+\omega)=x(t),\quad t\geqslant T,\\&m\leqslant x(t)\leqslant M,\quad t\geqslant T,\\&k(t,s)x(t)=g(x(s)),\quad t\geqslant T,\ t-\tau\leqslant s\leqslant t,\\&x(t)=1,\quad T-\tau\leqslant t\leqslant T\}.\end{aligned}$$

Define the operator $S:\Omega\to X$ as follows

$$(Sx)(t)=\begin{cases}\exp\left(-\int_T^t\int_{u-\tau}^{u} p(u-v)\frac{g(x(v))}{x(u)}\,dv\,du\right), & t\geqslant T,\\ 1, & T-\tau\leqslant t\leqslant T.\end{cases}$$