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A comparative study on the analytic solutions of fractional coupled sine–Gordon equations by using two reliable methods

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ABSTRACT

In this paper modified homotopy analysis method (MHAM) and homotopy perturbation transform method (HPTM) have been implemented for solving time fractional coupled sine–Gordon equations. We consider fractional coupled sine–Gordon equations which models one-dimensional nonlinear wave processes in two-component media. The results obtained by modified homotopy analysis method (MHAM) and homotopy perturbation transform method (HPTM) are then compared with the modified decomposition method (MDM). By using an initial value system, the numerical solutions of coupled sine–Gordon equations have been represented graphically. Here we obtain the solution of fractional coupled sine–Gordon (S–G) equations, which is obtained by replacing the time derivatives with a fractional derivatives of order $\alpha \in (1,2]$ and $\beta \in (1,2]$. The fractional derivatives here are described in Caputo sense.

1. Introduction

In this paper, the time-fractional order nonlinear coupled sine–Gordon equations [1] are considered in the following form:

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) = -\delta^2 \sin(u(x,t) - v(x,t)) \quad x \in \Re,\ t > 0 \tag{1.1}$$

$$D_t^{\beta} v(x,t) - c^2 v_{xx}(x,t) = \sin(u(x,t) - v(x,t)) \quad x \in \Re,\ t > 0 \tag{1.2}$$

where c is the ratio of the acoustic velocities of the components u and v. The dimensionless parameter δ^2 is equal to the ratio of masses of particles in the "lower" and the "upper" parts of the crystal in generalized Frenkel–Kontorova dislocation model [1–3]. Here, α, β are the parameters representing the order of fractional derivatives, which satisfy $m-1 < \alpha \leqslant m, n-1 < \beta \leqslant n$, and $t > 0$. When $\alpha = 2$ and $\beta = 2$, the fractional equation reduces to the classical coupled sine–Gordon equations.

Coupled sine–Gordon equations were introduced by Khusnutdinova and Pelinovsky [1]. The coupled sine–Gordon equations generalize the Frenkel–Kontorova dislocation model [2,3]. The Eqs. (1.1) and (1.2) with $c = 1$ were also proposed to describe the open states in DNA [4]. Khusnutdinova and Pelinovsky [1] have analyzed the linear and nonlinear wave process involving the exchange of energy between the two physical components of the system. They have obtained the linear solutions by considering $|u - v| \ll 1$, the exact nonlinear solutions for the case $c = 1$ and the weakly nonlinear solutions, for the

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general case, by means of asymptotic methods. In the recent past, notable researchers Saha Ray [5] and Kaya [6] had given the solutions of sine–Gordon equations by using the Modified Decomposition method (MDM).

In this paper, homotopy perturbation transform method [7–9] and homotopy analysis method [10–17] with modification have been applied for solving fractional coupled sine–Gordon (S–G) equations which play an important role in nonlinear physics. The homotopy perturbation transform method is a combined form of the Laplace transform method with the homotopy perturbation method. Currently, an analytical method for strongly nonlinear problems, namely the homotopy analysis method (HAM) [10–17] has been developed and successfully applied to many kinds of nonlinear problems in science and engineering. The homotopy analysis method (HAM) contains an auxiliary parameter $\hbar$ which provides us with a simple way to adjust the convergence region and rate of the series solution. Moreover, by means of the so-called $\hbar$-curve, it is easy to find the valid region of $\hbar$ to gain a convergent series solution. Thus, through HAM, explicit analytic solutions of nonlinear problems are possible to obtain. The above methods have been able to successfully find the solution without any discretization or restrictive assumptions and avoid the round-off errors.

The main objective of this paper is to employ two reliable independent analytical methods such as MHAM and HPTM for solving fractional coupled sine–Gordon equations. The capability, effectiveness and convenience of the methods have been established by obtaining the analytical solutions and comparing with that of MDM. In this regards, the solutions obtained by MDM method have been assumed as classical solutions for coupled S–G equations.

2. Mathematical preliminaries of fractional calculus

The fractional calculus involves different definitions of the fractional operators such as Riemann–Liouville fractional derivative, Caputo derivative, Riesz derivative and Grunwald–Letnikov fractional derivative. The fractional calculus has gained considerable importance during the past decades, mainly due to its applications in diverse fields of science and engineering. For the purpose of this paper the Caputo definition of fractional derivative will be used, with regard to the advantage of Caputo approach that the initial conditions for fractional differential equations with Caputo derivatives take on the traditional form as for integer-order differential equations.

Definition 2.1. A real function $f(t),\ t>0$, is said to be in the space $C_\mu;\mu\in\Re$ if there exists a real number $p(>\mu)$, such that $f(t)=t^pf_1(t)$, where $f_1(t)\in C[0,\infty)$, and it is said to be in the space $C_\mu^{(m)},f^{(m)}\in C_m, m\in N$. The Riemann–Liouville fractional integral operator is defined as follows:

Definition 2.2 (*Riemann–Liouville integral*). The most frequently encountered definition of an integral of fractional order is the Riemann–Liouville integral, in which the fractional integral of order $\alpha(>0)$ is defined as [18,19]

$$J_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\quad t>0,\quad \alpha\in\Re^+ \tag{2.2.1}$$

$$J_t^0f(t)=f(t)$$

where $\Re^+$ is the set of positive real numbers.

Properties of the operator J^α can be found in [18,19] and we mention only the following: for $f\in C_\mu,\mu\geqslant 1,\alpha,\beta\geqslant 0$ and $\gamma\geqslant 1$, we have

$$\text{(i) } J^\alpha J^\beta f(t)=J^{\alpha+\beta}f(t)$$
$$\text{(ii) } J^\alpha J^\beta f(t)=J^\beta J^\alpha f(t)$$
$$\text{(iii) } J^\alpha t^\gamma=\frac{\Gamma(\gamma+1)t^{\gamma+\alpha}}{\Gamma(\alpha+\gamma+1)} \tag{2.2.2}$$

Lemma 2.3. *If $m-1<\alpha\leqslant m, m\in N$ and $f\in C_\mu^m,\mu\geqslant 1$, then*

$$D^\alpha J^\alpha f(t)=f(t)\quad \textit{and}$$

$$J^\alpha D^\alpha f(t)=f(t)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{t^k}{k!},\quad t>0$$

Definition 2.4 (*Caputo fractional derivative*). The fractional derivative introduced by Caputo, is called Caputo fractional derivative. The fractional derivative of $f(t)$ in the Caputo sense is defined by [18,19]

$$D_t^\alpha f(t)=J^{m-\alpha}f^{(m)}(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau,\quad \text{for } m-1<\alpha\leqslant m,\ m\in N,\ t>0,\ f\in C_{-1}^m \tag{2.4.1}$$