Applied Mathematics and Computation 253 (2015) 102–115

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Vector-valued Gabor frames associated with periodic subsets of the real line ☆

Yun-Zhang Li [a,*], Yan Zhang [b]

[a] College of Applied Sciences, Beijing University of Technology, Beijing 100124, PR China
[b] School of Mathematics and Information Science, Beifang University of Nationalities, Yinchuan 750021, PR China

ARTICLE INFO

Keywords:
Frame
Gabor frame
Vector-valued Gabor frame
Gabor dual
Zak transform

ABSTRACT

The notion of vector-valued frame (also called superframe) was first introduced by Balan in the context of multiplexing. It has significant applications in mobile communication, satellite communication, and computer area network. For vector-valued Gabor analysis, existent literatures mostly focus on $L^2(\mathbb{R}, \mathbb{C}^L)$ instead of its subspace. Let $a > 0$, and S be an $a\mathbb{Z}$-periodic measurable set in $\mathbb{R}$ (i.e. $S + a\mathbb{Z} = S$). This paper addresses Gabor frames in $L^2(S, \mathbb{C}^L)$ with rational time–frequency product. They can model vector-valued signals to appear periodically but intermittently. And the projections of Gabor frames in $L^2(\mathbb{R}, \mathbb{C}^L)$ onto $L^2(S, \mathbb{C}^L)$ cannot cover all Gabor frames in $L^2(S, \mathbb{C}^L)$ if $S \neq \mathbb{R}$. By introducing a suitable Zak transform matrix, we characterize completeness and frame condition of Gabor systems, obtain a necessary and sufficient condition on Gabor duals of type I (resp. II) for a general Gabor frame, and establish a parametrization expression of Gabor duals of type I (resp. II). All our conclusions are closely related to corresponding Zak transform matrices. This allows us to easily realize these conclusions by designing the corresponding matrix-valued functions. An example theorem is also presented to illustrate the efficiency of our method.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. An at most countable sequence $\{h_i\}_{i\in\mathcal{I}}$ in $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{i\in\mathcal{I}} |\langle f, h_i\rangle|^2 \leqslant B\|f\|^2 \tag{1.1}$$

for $f \in \mathcal{H}$, where $A,\ B$ are called *frame bounds*; it is called a *tight frame* (*Parseval frame*) if $A = B$ ($A = B = 1$) in (1.1); and a *Bessel sequence* in $\mathcal{H}$ if the right-hand side inequality in (1.1) holds. A frame for $\mathcal{H}$ is called a *Riesz basis* if it ceases to be a frame whenever any one of its elements is removed. Given two Bessel sequences $\{g_i\}_{i\in\mathcal{I}}$ and $\{h_i\}_{i\in\mathcal{I}}$ in $\mathcal{H}$, define the operator $\mathcal{S}_{h,g} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{S}_{h,g}f = \sum_{i\in\mathcal{I}} \langle f, h_i\rangle g_i \tag{1.2}$$

☆ Supported by the National Natural Science Foundation of China (Grant No. 11271037) and Beijing Natural Science Foundation (Grant No. 1122008).
* Corresponding author.
E-mail addresses: yzlee@bjut.edu.cn (Y.-Z. Li), yzhangbun@163.com (Y. Zhang).

http://dx.doi.org/10.1016/j.amc.2014.12.046
0096-3003/

for $f \in \mathcal{H}$. Then $\mathcal{S}_{h,g}$ is a bounded operator on $\mathcal{H}$. Let $\{g_i\}_{i\in\mathcal{I}}$ be a frame for $\mathcal{H}$. A frame $\{h_i\}_{i\in\mathcal{I}}$ is called a *dual* of $\{g_i\}_{i\in\mathcal{I}}$ if $\mathcal{S}_{h,g} = I$ on $\mathcal{H}$, where I denotes the identity operator. It is well-known that, for two Bessel sequences $\{g_i\}_{i\in\mathcal{I}}$ and $\{h_i\}_{i\in\mathcal{I}}$ in $\mathcal{H}$, whenever $\mathcal{S}_{h,g} = I$ on $\mathcal{H}$, they are both frames for $\mathcal{H}$ and are duals of each other. If $g_i = h_i$ in (1.2) and $\{g_i\}_{i\in\mathcal{I}}$ is a frame for $\mathcal{H}$ with frame bounds A and B, it is also well-known that $\mathcal{S}_{g,g}$ is bounded and invertible, that $\{\mathcal{S}_{g,g}^{-1}g_i\}_{i\in\mathcal{I}}$ is also a frame for $\mathcal{H}$ with frame bounds B^{-1} and A^{-1}, and a dual of $\{g_i\}_{i\in\mathcal{I}}$, which is the so-called *canonical dual*. The fundamentals of frames can be found in [9,10,20,31].

Given a positive integer L, let $L^2(\mathbb{R}, \mathbb{C}^L)$ be the vector-valued Hilbert space $L^2(\mathbb{R}, \mathbb{C}^L)$ endowed with the inner product defined by

$$\langle \mathbf{f}, \mathbf{h}\rangle = \sum_{l=1}^{L}\int_{\mathbb{R}} f_l(x)\overline{h_l(x)}dx \quad \text{for} \quad \mathbf{f} = (f_1, f_2, \ldots, f_L), \quad \mathbf{h} = (h_1, h_2, \ldots, h_L) \in L^2(\mathbb{R}, \mathbb{C}^L).$$

Obviously, it is exactly the direct sum Hilbert space $\bigoplus_{l=1}^{L} L^2(\mathbb{R})$. In what follows, for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$ and $1 \leqslant l \leqslant L$, we always denote by f_l its l-th component. For $a, b > 0$ and $\mathbf{g} \in L^2(\mathbb{R}, \mathbb{C}^L)$, we define *Gabor system* $G(\mathbf{g}, a, b)$ by

$$G(\mathbf{g}, a, b) = \{E_{mb}T_{na}\mathbf{g} : m, n \in \mathbb{Z}\}, \tag{1.3}$$

where

$$E_{mb}T_{na}\mathbf{g} = \left(e^{2\pi imb\cdot}g_1(\cdot - na),\, e^{2\pi imb\cdot}g_2(\cdot - na),\, \ldots,\, e^{2\pi imb\cdot}g_L(\cdot - na)\right).$$

We also call it vector-valued Gabor system since L is not necessarily 1. When $L = 1$, it is the usual Gabor system in $L^2(\mathbb{R})$ and called scalar-valued Gabor system in contrast to a general L. A set S in $\mathbb{R}$ with positive measure is said to be $a\mathbb{Z}$-*periodic* if $S + an = S$ for $n \in \mathbb{Z}$. For such S, we denote by $L^2(S, \mathbb{C}^L)$ the closed subspace of $L^2(\mathbb{R}, \mathbb{C}^L)$ of the form

$$L^2(S, \mathbb{C}^L) = \left\{\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L) : \mathbf{f} = 0 \text{ on } \mathbb{R}\setminus S\right\}.$$

This paper addresses Gabor analysis on $L^2(S, \mathbb{C}^L)$.

Vector-valued frame is also called superframe. It was introduced in [4] under the setting of general Hilbert spaces by Balan in the context of "multiplexing", which has been widely used in mobile communication network, satellite communication network and computer area network. In recent years, vector-valued wavelet and Gabor frames in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ have interested some mathematicians and engineering specialists (see [2,3,5,11–13,21,22,24,25] and references therein), and in [29,30] vector-valued analysis also occurred as a technical tool in the study of ordinary frames. Let us first recall some related works.

• Vector-valued Gabor frames for the whole space $L^2(\mathbb{R}, \mathbb{C}^L)$.

Führ in [13] derived frame bound estimates for vector-valued Gabor system in $L^2(\mathbb{R}, \mathbb{C}^L)$ with window functions belonging to Schwartz space, and obtained estimates for the window $\mathbf{h} = (h_0, \quad h_1, \quad \ldots, \quad h_L) \in L^2(\mathbb{R}, \mathbb{C}^{L+1})$ composed of the first $L+1$ Hermite functions. Gröchenig and Lyubarskii in [21] characterized all lattices $\Lambda \subset \mathbb{R}^2$ such that the Gabor system $\{E_{\lambda_2}T_{\lambda_1}\mathbf{h} : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^L)$. Abreu [1] gave a simple proof of this characterization. It has the advantage of also characterizing all lattices $\Lambda \subset \mathbb{R}^2$ such that $\{E_{\lambda_2}T_{\lambda_1}\mathbf{h} : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a Riesz sequence in $L^2(\mathbb{R}, \mathbb{C}^L)$. Also observe that Brekke and Seip in [7] characterized sets generating multi-window Gabor frames (resp. Riesz sequences) with Hermite functions. For general vector-valued Gabor systems, necessary density conditions were studied in [3] by Balan. For vector-valued Gabor systems with rational time–frequency lattices, a sufficient and necessary density condition was obtained in [25] by Li and Han. And a Zak transform matrix method was developed in [27] by Li and Zhou. There authors characterized complete vector-valued Gabor systems and Gabor frames, and obtained a parametrization of all its Gabor duals of a general vector-valued Gabor frame.

• Scalar-valued subspace Gabor frames.

The theory of subspace Gabor frames includes two aspects. One is to ask whether $G(g, a, b)$ is a frame for its closed linear span for given $g \in L^2(\mathbb{R})$ and $a, b > 0$, and [6,8,14–16,32] belong to this. The other is, given $a, b > 0$ and an $a\mathbb{Z}$-periodic set S in $\mathbb{R}$, to find g such that $G(g, a, b)$ is a frame for $L^2(S)$. See [17–19,23,28] for details. Gabor analysis on $L^2(S)$ interests us because of the following reasons:

•• From the perspective of application. Gabor systems on $L^2(S)$ can model a situation where a signal is known to appear periodically but intermittently, and one would try to perform Gabor analysis for the signal in the most efficient way possible while still preserving all the features of the observed data. Although one can think of the signal as existing for all time and do the analysis in the usual way, this is not the optimal way to proceed if the signal is only emitted for very short periods of time.

•• From the perspective of theory. The $a\mathbb{Z}$-periodicity of S is a natural requirement since one can prove that S must be $a\mathbb{Z}$-periodic if $L^2(S)$ admits a complete Gabor system. The projections of Gabor frames in $L^2(\mathbb{R})$ onto $L^2(S)$ cannot cover all Gabor frames in $L^2(S)$. Indeed, let $ab \leqslant 1$, and S be an $a\mathbb{Z}$-periodic measurable subset of $\mathbb{R}$ with positive measure. It is easy to check that, if $G(g, a, b)$ is a frame for $L^2(\mathbb{R})$, then its projection $G(g\chi_S, a, b)$ onto $L^2(S)$ is a frame for $L^2(S)$, where χ_S is the characteristic function of S. However, when $ab > 1$ and $S \neq \mathbb{R}$, $G(g, a, b)$ cannot be a frame in $L^2(\mathbb{R})$ for any $g \in L^2(\mathbb{R})$, while it is possible that there exists some g such that $G(g, a, b)$ is a frame for $L^2(S)$. In addition, Theorems 2.7, 2.12, 3.3, 4.2 and Corollaries 2.13, 4.3 in [18] show that there exist significant differences between Gabor analysis on $L^2(S)$ and one on $L^2(\mathbb{R})$.