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Arnoldi methods for image deblurring with anti-reflective boundary conditions

Marco Donatelli [a,*], David Martin [b], Lothar Reichel [b]

[a] Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria, Como 22100, Italy
[b] Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

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ABSTRACT

Image deblurring with anti-reflective boundary conditions and a non-symmetric point spread function is considered. Several iterative methods based on Krylov subspace projections, as well as Arnoldi–Tikhonov regularization methods, with reblurring right or left preconditioners are compared. The aim of the preconditioner is not to accelerate the convergence, but to improve the quality of the computed solution and to increase the robustness of the regularization method. Right preconditioning in conjunction with methods based on the Arnoldi process are found to be robust and give high-quality restorations. In particular, when the observed image is contaminated by motion blur, our new method is much more effective than other approaches described in the literature, such as range restricted Arnoldi methods and the conjugate gradient method applied to the normal equations (implemented with the reblurring approach).

1. Introduction

We consider the restoration of blurred and noise-corrupted images, where the blurring is modeled by a convolution. In two space-dimensions this problem can be formulated as an integral equation of the form

$$g(\boldsymbol{x}) = [Kf](\boldsymbol{x}) + \nu(\boldsymbol{x}) = \int_{\mathbb{R}^2} h(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \nu(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2, \tag{1}$$

where f represents the (unavailable) true image, h the space invariant point-spread function (PSF) with compact support, ν random noise, and g the (available) blurred and noise-corrupted image. Thus, f and g are real-values nonnegative functions that yield the gray-scale values of the images; see, e.g. [3] for a discussion of this model.

Discretization of the above integral equation at equidistant nodes yields

$$g_i = \sum_{j\in\mathbb{Z}^2} h_{i-j} f_j + \nu_i, \quad i \in \mathbb{Z}^2, \tag{2}$$

where the entries of the discrete images $\boldsymbol{g} = [g_i]$ and $\boldsymbol{f} = [f_j]$ represent the light intensity at each pixel and $\boldsymbol{\nu} = [\nu_i]$ models the noise-contamination at these pixels.

* Corresponding author.
E-mail addresses: marco.donatelli@uninsubria.it (M. Donatelli), dmarti49@kent.edu (D. Martin), reichel@math.kent.edu (L. Reichel).

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We are interested in determining an accurate approximation of the true image $\boldsymbol{f}$ in the finite field of view (FOV) corresponding to $i \in [1, n]^2$, given $\boldsymbol{h} = [h_i]$, distributional information about $\boldsymbol{v}$, and the blurred image $\boldsymbol{g}$ in the same FOV. The FOV is assumed to be square only for notational simplicity.

The linear system of equations defined by (2) with i restricted to $[1, n]^2$ is underdetermined when there are nonvanishing coefficients h_i with $i \neq 0$, since then there are n^2 constraints, while the number of unknowns required to specify the equations is larger. A meaningful solution of this system can be determined in several ways. For instance, one may determine the solution of minimal Euclidean norm of the underdetermined system; see [4,28] for discussions of this approach. Alternatively, one can determine a linear system of equations with a square matrix,

$$A\boldsymbol{f} = \boldsymbol{g}, \qquad A \in \mathbb{R}^{n^2 \times n^2}, \quad \boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^{n^2}, \tag{3}$$

by imposing boundary conditions (BC), where the f_j-values in (2) at pixels outside the FOV are assumed to be certain linear combinations of values inside the FOV. Popular boundary conditions include zero Dirichlet boundary conditions (ZDBC), periodic boundary conditions (PBC), reflective boundary conditions (RBC) discussed in [23], and anti-reflective boundary conditions (ARBC) proposed in [27]. An illustration of images that satisfy these boundary conditions can be found in, e.g. [18]. Further details can be found in Section 2.

We say that an image $\boldsymbol{f}$ is discontinuous at the boundary, if the corresponding continuous function f has to have a discontinuity or a steep gradient in order for $\boldsymbol{f}$ to be a discretization of f.[1] For generic images, both ZDBC and PBC usually introduce discontinuities in the image $\boldsymbol{g}$ at the boundary, often resulting in artifacts such as "ringing" near the boundary of the restored image $\boldsymbol{f}$. Generally, such artifacts can be reduced significantly by the use of RBC or ARBC, since both these boundary conditions impose continuity of the extended image at the boundary. Further, ARBC tend to reduce artifacts near the boundary of the restored image $\boldsymbol{f}$ more than RBC, because the former boundary conditions also impose continuity of the normal derivative of the extended image at the boundary. Similarly as continuity, the notion of a normal derivative has to be suitably interpreted.

When the PSF h is quadrantally symmetric, i.e., when

$$h(x_1, x_2) = h(|x_1|, |x_2|) \quad \forall \boldsymbol{x} = [x_1, x_2]^T \in \Omega$$

such as the PSF associated with symmetric Gaussian blur, there are fast transforms for diagonalizing the matrix A in (3) when RBC or ARBC are imposed. These transforms can be applied to devise fast filtering methods for the approximate solution of (3); see [1,8,23]. For many more general PSFs of interest in applications, fast filtering methods are not available. For instance, let the matrix A be determined by a nonsymmetric PSF h that models motion blur with RBC or ARBC. Then there are fast algorithms for the evaluation of matrix–vector products with A, but there is no known direct algorithm for the fast approximate solution of the associated linear system of equations (3). Therefore, when using RBC or ARBC with this kind of PSF, one generally resorts to iterative methods for the restoration of large images.

The adjoint of the convolution operator in (1) is the correlation operator

$$[K^* f](\boldsymbol{x}) = \int_{\mathbb{R}^2} h(\boldsymbol{y} - \boldsymbol{x}) f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{4}$$

where we have used the fact that h is real-valued. In [9], the authors propose to replace the system (3) by

$$A' A \boldsymbol{f} = A' \boldsymbol{g}, \tag{5}$$

when RBC or ARBC are imposed and the PSF is quite general. The matrix A' is obtained from the discretization of (4) with the same boundary conditions as for (3), i.e., from the PSF rotated 180°. The system (5) offers an alternative to the normal equations,

$$A^T A \boldsymbol{f} = A^T \boldsymbol{g}, \tag{6}$$

when the evaluation of matrix–vector products with the transpose A^T of the matrix A is complicated or time-consuming. This situation arises when the matrix A is not explicitly stored and has a structure that can be utilized when evaluating matrix–vector products efficiently, while the structure of A^T is more difficult to exploit. This is the case, for instance, when A is defined by a convolution in two or more space-dimension and ARBC are imposed. We remark that in one space-dimension, matrix–vector products with A and A^T can be evaluated about equally rapidly. Further comments on the difficulties of using A^T are provided in Section 2.

The solution of (5) has been shown to reduce boundary artifacts when the PSF is a symmetric convolution; see [11] and Section 5 for illustrations that compare restorations determined by solving (5) and (6), as well as the unpreconditioned system (3). The solution of (5) is found to give the best restorations. Note that when either ZDBC or PBC are imposed, we have $A' = A^T$ and, hence, Eq. (5) agrees with the normal equations (6).

The standard CGLS method is an efficient implementation of the conjugate gradient method applied to the normal equations (6). The method applies a three-term recursion, which is analogous to the one used by the symmetric Lanczos process, to reduce a large symmetric matrix to a small symmetric tridiagonal matrix; see [5] for details. In [9], the recursion formulas for CGLS are applied to the solution of (5) when ARBC are imposed. Thus, the iterative method for the solution of (5) is

[1] We may, for instance, consider f to be a piecewise linear function that interpolates the pixel values $\boldsymbol{f}$.