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Period adding structure in a 2D discontinuous model of economic growth

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ABSTRACT

We study the dynamics of a growth model formulated in the tradition of Kaldor and Pasinetti where the accumulation of the ratio capital/workers is regulated by a two-dimensional discontinuous map with triangular structure. We determine analytically the border collision bifurcation boundaries of periodicity regions related to attracting cycles, showing that in a two-dimensional parameter plane these regions are organized in the period adding structure. We show that the cascade of flip bifurcations in the base one-dimensional map corresponds for the two-dimensional map to a sequence of pitchfork and flip bifurcations for cycles of even and odd periods, respectively.

1. Introduction

An important limitation of neoclassical models of economic growth consists in exclusively predicting for the long run a monotonic convergence to a steady state for both output and capital per capita. A lot of efforts have been made by economists to conceive the possibility of endogenous growth cycles. If we only focus on the so-called Solow–Swan growth model, it has been proved, for instance, that complicated growth paths may arise by considering two sectors instead of one, combined with certain levels of discounting (see [1,2]). Other researchers have introduced economically founded nonlinearities into the models in order to make unstable the steady state or even increase the number of stationary states, with the possibility of complicated dynamics. A pioneer in this strand of research is Richard Day, who in a pair of seminal papers [3,4] improved the Solow model by introducing nonlinearities leading to irregular growth cycles. One of these possibilities consists in replacing the unrealistic hypothesis of exponential growth of the labor force with a more realistic bounded growth such as the logistic one. This alternative formalization of the labor force growth rate has been successfully implemented into the classical Ramsey growth model [5–10], the Solow–Swan framework [11] and the Kaldor–Pasinetti model with differential savings [12].

Endogenous fluctuations of the growth path can also be generated by the introduction of discontinuities in an otherwise classical framework. From a mathematical point of view a discontinuity, like a nonlinearity, may cause the emergence of complex dynamics (cycles and chaos), however routes to such dynamics are quite different. To our knowledge only Böhm and Kaas [13] and Tramontana et al. [14] give examples of investigations of the role of a discontinuity in a classical growth

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model.[1] They move from a Kaldor–Pasinetti model with differential savings and introduce a discontinuity through a Leontief production function, showing that growth cycles are a typical outcome under these assumptions.

Recently, in [17] a growth model is built which combines the nonlinearity of the logistic growth of the labor force with the discontinuity arising from the assumption of Leontief technology (as in [13,14]). The authors explain the economic meaning of these assumptions and show some numerical simulations with interesting dynamic outcomes.

The aim of the present paper is to investigate the bifurcations occurring in the model proposed in [17], which is described by a two-dimensional (2D henceforth in short) discontinuous triangular map at the base of which there is the well known logistic map. We show that quite complicated growth paths may emerge and some of the observed phenomena are qualitatively different from those arising in models with nonlinearities and discontinuities taken separately. In particular, bistability is proved to occur in the present 2D map. Moreover, we show that the periodicity regions related to stable cycles are organized in the period adding structure. Note that such a structure is characteristic for a class of one-dimensional (1D for short) discontinuous piecewise monotone maps (see, among others, [18–20,14]). It has been shown that the period adding structure can be observed also in 1D maps with two discontinuities [21], and in continuous 1D maps with two border points [22]. A first example of the period adding structure in 2D discontinuous maps is provided in [23]. In the present paper we explain how to analytically obtain the equations of the border collision bifurcation[2] (BCB for short) boundaries of the periodicity regions in the parameter space.

The paper is organized as follows. In Section 2 we introduce the economic assumptions that permit to obtain a 2D discontinuous map governing the dynamics of the capital accumulation. Some preliminary results are presented in Section 3. Different bifurcation scenarios originating by the map are studied in the subsequent three sections. In particular, in Section 4 we describe the asymptotic dynamics occurring in a 1D piecewise linear map with one discontinuity, which is a restriction of the 2D map to an invariant straight line (layer) associated with the fixed point of the base map. A first period doubling bifurcation of this fixed point leads to two cyclic layers (associated with the 2-cycle of the logistic map) on which the restriction of the 2D map is a 1D piecewise linear map with at most three discontinuity points. The asymptotic dynamics of this map is studied in Section 5. These 1D maps allow us to explain the existence of a period adding structure in the parameter space of the 2D map, and to analytically determine the equations of the border collision bifurcations boundaries of the periodicity regions presented in Appendix. In Section 6 we prove the occurrence of bistability associated with flip bifurcations in the base map, which for the 2D map are pitchfork bifurcations (leading thus to bistability) for the cycles of even periods or flip bifurcations for the cycles of odd periods. In Section 7 we propose some final observations.

2. The model

We consider a classic discrete time one sector Solow–Swan growth model enriched by the following additional assumptions:

- two groups of agents, workers and shareholders (see [24–26]) are characterized by constant but different saving propensities, usually denoted as s_w and s_r in the cited literature, and here denoted in a concise form as w and r, with $0 \leqslant w \leqslant r \leqslant 1$;
- technology is characterized by a Leontief production function (see [13]):

$$f(k) = \min(ak, b) + c, \tag{1}$$

where k denotes capital per worker, and $a,\ b,\ c$ are positive technical parameters;
- a logistic labor force growth rate (n).

The usual way of determining the wage rate W is the following:

$$W(k) = f(k) - kf'(k), \tag{2}$$

where the marginal product $f'(k)$ is what shareholders gather while $kf'(k)$ is the capital income per worker.

Considering a one-period production lag and a capital depreciation rate $0 < \delta \leq 1$ we get the following equation that regulates the growth path of the capital accumulation:

$$k_{t+1} = \frac{1}{1+n_t}\left[(1-\delta)k_t + wW(k_t) + rk_tf'(k_t)\right]. \tag{3}$$

Substituting the wage rate Eq. (2), using the Leontief production function (1) and the assumption of logistic capital force growth rate, we finally obtain the following 2D discontinuous map:

[1] Actually we should also mention the growth model studied by Matsuyama [15] and Gardini et al. [16] where combining Solow and Romer models a piecewise smooth map is obtained. However, the map is continuous, and this leads to dynamics quite different from those occurring in a discontinuous one.

[2] Recall that in discontinuous maps the BCB of a cycle occurs when one of its periodic points collides with the border of the existence region of the cycle, causing its disappearance.