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Gibbs phenomenon in the Hermite interpolation on the circle ☆

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ABSTRACT

Hermite–Fejér interpolation problems on the unit circle and bounded interval are usually studied in relation with continuous functions. There are few references concerning these problems for functions with discontinuities. Thus the aim of this paper is to describe the behavior of the Hermite–Fejér and Hermite interpolants for piecewise continuous functions on the unit circle, analyzing the corresponding Gibbs phenomenon near the discontinuities and providing the asymptotic amplitude of the Gibbs height.

1. Introduction

It is well known that Lagrange polynomial interpolants corresponding to continuous functions on bounded intervals and general nodal systems, have an irregular behavior in relation with the convergence. To obtain better results of convergence it is necessary to impose some condition to the function, as for example about its modulus of continuity. Alternatively, the properties of convergence can be improved if one interpolates using values for the derivatives of the polynomial interpolants. One way to do that consists in taking null values for the derivatives of the interpolants at the nodes, which is the so called Hermite–Fejér interpolation. In this case it is guaranteed the uniform convergence. Another way of interpolating consists in taking non vanishing values for the derivatives of the polynomial interpolants. These values, which may be or not related with the function, must satisfy some additional conditions in order to obtain uniform convergence. The same behavior holds for the interpolation on the unit circle and the corresponding results for equally spaced nodal systems can be seen in [8,3]. One of the differences to take into account is the use of Laurent polynomials instead of algebraic polynomials due to Stone–Weiertrass theorem. Thus Hermite–Fejér and Hermite interpolation are very suitable for interpolating continuous functions which do not need to be differentiable. Of course, in case of smooth functions if we have more information, Hermite interpolation is very useful.

Perhaps due to the reasons given before, Hermite–Fejér interpolation problems on the unit circle and bounded interval are usually studied in relation with continuous functions. Nevertheless there are some references concerning these problems for functions with discontinuities, piecewise continuous functions and piecewise smooth functions. We highlight a paper of Poghosyan [15] and two interesting papers related with the local behavior of the interpolants due to Bojanic and Cheng [4,5].

Related problems, such as Fourier series, in which Gibbs phenomenon appears, have been thoroughly studied for the case of discontinuities. The same situation has also been studied for the Lagrange trigonometric interpolation problem. Several results concerning the limit function for equidistant nodes as well as a study of the Gibbs phenomenon appear in the papers

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of Helmberg [10,11] and Helmberg and Wagner [12]. Concerning Lagrange interpolation on the bounded interval with Chebyshev nodes, the problem has been analyzed in the book of Trefethen [16]. Indeed, one of the chapters of [16] is devoted to gather results about Lagrange interpolation for piecewise continuous functions on intervals, including the description of Gibbs phenomenon and the determination of the height of the jump.

Our aim in this paper is to describe the behavior of the Hermite–Fejér and Hermite interpolants for piecewise continuous functions and/or piecewise smooth functions on the circle and to analyze the corresponding Gibbs phenomenon near the discontinuities, providing detailed proofs of our results and the asymptotic amplitude of the Gibbs height.

The organization of the paper is the following. In Section 2 we pose the Hermite interpolation problem to be solved and we present several expressions for the corresponding Hermite interpolation polynomials. Section 3 is devoted to study the behavior, on compact subsets without discontinuity points, of the Hermite–Fejér interpolation polynomials for piecewise continuous functions on the unit circle $\mathbb{T}$. In Section 4 we analyze the behavior of the Hermite–Fejér interpolants corresponding to the characteristic function of an arc contained in $\mathbb{T}$. The behavior is described for every point in $\mathbb{T}$ and it is complemented with some numerical experiments. Finally, the last section is devoted to study the convergence on $\mathbb{T}$ of the Hermite–Fejér and Hermite interpolants for piecewise continuous functions and/or piecewise smooth functions. We also remark some differences that we have detected between the Gibbs phenomenon in the Hermite interpolation and in the other processes of Fourier series and Lagrange interpolation, in which this phenomenon appears.

2. Preliminaries

We recall that the Hermite interpolation problem on the unit circle with nodal system $\{\alpha_j\}_{j=0}^{n-1}$ constituted by the n-roots of a complex number λ, with $|\lambda| = 1$, consists in obtaining a Laurent polynomial

$$\mathcal{H}_{-n,n-1} \in \Lambda_{-n,n-1} = \operatorname{span}\{z^k : -n \leqslant k \leqslant n-1\},$$

that satisfies the interpolation conditions

$$\mathcal{H}_{-n,n-1}(\alpha_j) = u_j \quad \text{and} \quad \mathcal{H}'_{-n,n-1}(\alpha_j) = v_j, \quad \text{for } j = 0, \ldots, n-1, \tag{1}$$

where $\{u_j\}_{j=0}^{n-1}$ and $\{v_j\}_{j=0}^{n-1}$ are fixed complex values. It is clear that α_j, u_j and v_j also depend on n; however, for simplicity, we omit the parameter n.

The situation corresponding to $v_j = 0$, $\forall j = 0, \ldots, n-1$, is called Hermite–Fejér (H–F) interpolation and the polynomial is denoted by $\mathcal{HF}_{-n,n-1}(z)$.

If f is a function defined on $\mathbb{T}$ and $u_j = f(\alpha_j)$, $\forall j = 0, \ldots, n-1$, the corresponding H–F interpolation polynomial is denoted by $\mathcal{HF}_{-n,n-1}(f, z)$.

In a more general setting the problem can be posed as follows. If $p(n)$ and $q(n)$ are two nondecreasing sequences of non-negative integers such that $p(n) + q(n) = 2n - 1$ for $n \geqslant 1$, find the unique Laurent polynomial $\mathcal{H}_{-p(n),q(n)} \in \Lambda_{-p(n),q(n)} = \operatorname{span}\{z^k : -p(n) \leqslant k \leqslant q(n)\}$ satisfying the interpolation conditions

$$\mathcal{H}_{-p(n),q(n)}(\alpha_j) = u_j \quad \text{and} \quad \mathcal{H}'_{-p(n),q(n)}(\alpha_j) = v_j, \quad \text{for } j = 0, \ldots, n-1, \tag{2}$$

where $\{u_j\}_{j=0}^{n-1}$ and $\{v_j\}_{j=0}^{n-1}$ are prefixed complex numbers. To obtain results of convergence one has also to assume that $|\, p(n) - n \,|$ is bounded.

For simplicity and without loss of generality, in the sequel we consider the case corresponding to $p(n) = -n$ and $q(n) = n - 1$. The problem stated by (2) can be solved in a similar way.

It is well known that the interpolation polynomial $\mathcal{H}_{-n,n-1}$ can be computed by means of different expressions. One of these formulas is the classical one, given in terms of the fundamental polynomials of Hermite interpolation in the following way

$$\mathcal{H}_{-n,n-1} = \sum_{j=0}^{n-1} \left(u_j \mathcal{A}_{j,n} + v_j \mathcal{B}_{j,n}\right), \tag{3}$$

where $\mathcal{A}_{j,n}$ and $\mathcal{B}_{j,n}$, for $j = 0, \ldots, n-1$, are the fundamental polynomials of Hermite interpolation of the first and the second kind respectively and they are characterized by

$$\mathcal{A}_{j,n}(\alpha_i) = \delta_{i,j}, \quad \mathcal{A}'_{j,n}(\alpha_i) = 0, \quad \forall i = 0, \ldots, n-1,$$

$$\mathcal{B}_{j,n}(\alpha_i) = 0, \quad \mathcal{B}'_{j,n}(\alpha_i) = \delta_{i,j}, \quad \forall i = 0, \ldots, n-1.$$

In [8] it can be seen the following suitable expressions for these fundamental polynomials:

$$A_{j,n}(z) = \frac{(z^n - \lambda)^2}{z^n n^2 \lambda} \frac{\alpha_j z}{(z - \alpha_j)^2} \quad \text{and} \quad B_{j,n}(z) = \frac{\alpha_j^2 (z^n - \lambda)^2}{z^n n^2 \lambda (z - \alpha_j)} \tag{4}$$

for $j = 0, \ldots, n-1$.