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Some best approximation formulas and inequalities for the Wallis ratio

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ABSTRACT

In the paper, the authors establish some best approximation formulas and inequalities for the Wallis ratio. These formulas and inequalities improve an approximation formula and a double inequality for the Wallis ratio presented in 2013 by three mathematicians.

1. Introduction

The Wallis ratio is defined as

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)},$$

where Γ is the classical Euler gamma function which may be defined by

$$\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}\mathrm{d}u, \quad \Re(z) > 0. \tag{1.1}$$

The study and applications of W_n have a long history, a large amount of literature, and a lot of new results. For detailed information, please refer to the papers [1,4,18,19,22], related texts in the survey articles [17,20,21] and references cited therein. Recently, Guo, Xu, and Qi proved in [5] that the double inequality

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n} < W_n \leqslant \frac{4}{3}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n}, \tag{1.2}$$

for $n \geq 2$ is valid and sharp in the sense that the constants $\sqrt{\frac{e}{\pi}}$, and $\frac{4}{3}$ in (1.2) are best possible. They also proposed in [5] the approximation formula

$$W_n \sim \chi_n := \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n}, \quad n \to \infty. \tag{1.3}$$

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The sharpness of the double inequality (1.2) was proved in [5] basing on the variation of a function which decreases on $[2,\infty)$ from $\frac{4}{3}$ to $\sqrt{\frac{e}{\pi}}$. As a consequence, the right-hand side of (1.2) becomes weak for large values of n. Moreover, if we are interested to estimating W_n when n approaches infinity, then the constant $\sqrt{\frac{e}{\pi}}$ should be chosen and inequalities using $\sqrt{\frac{e}{\pi}}$, are welcome.

The aim of this paper is to improve the double inequality (1.2) and the approximation formula (1.3).

2. A lemma

For improving the double inequality (1.2) and the approximation formula (1.3), we need the following lemma.

Lemma 2.1 [12, Lemma 1.1]. *If the sequence $\{\omega_n : n \in \mathbb{N}\}$ converges to 0 and*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = \ell \in \mathbb{R}, \tag{2.1}$$

for $k > 1$, then

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{\ell}{k-1}. \tag{2.2}$$

Remark 2.1. Lemma 2.1 was first established in [15] and has been effectively applied in many papers such as [2,3,6–11,13,14,16].

3. A best approximation formula

With the help of Lemma 2.1, we first provide a best approximation formula of the Wallis ratio W_n.

Theorem 3.1. *The approximation formula*

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}}, \quad n \to \infty, \tag{3.1}$$

is the best approximation of the form

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n+a}}{n}, \quad n \to \infty, \tag{3.2}$$

where a is a real parameter.

Proof. Define $z_n(a)$ by

$$W_n = \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n+a}}{n} \exp z_n(a), \quad n \geqslant 1.$$

It is not difficult to see that $z_n(a) \to 0$ as $n \to \infty$. A direct computation gives

$$z_n(a) - z_{n+1}(a) = -\frac{a}{2n^2} + \left(\frac{1}{2}a + \frac{1}{2}a^2 + \frac{1}{12}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$\lim_{n\to\infty}\{n^2[z_n(a) - z_{n+1}(a)]\} = -\frac{a}{2}.$$

Making use of Lemma 2.1, we immediately see that the sequence $\{z_n(a) : n \in \mathbb{N}\}$ converges fastest only when $a = 0$. The proof of Theorem 3.1 is complete. □

Remark 3.1. The approximation formula (3.1) is an improvement of (1.3), since the approximation formula (1.3) is the special case $a = -1$ in (3.2).

4. An asymptotic series associated to (3.1)

In this section, by discovering an asymptotic series and a single-sided inequality for the Wallis ratio, we further generalize the approximation formula (3.1) and improve the left-hand side of the double inequality (1.2).