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Pairwise comparisons simplified

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ABSTRACT

This study examines the notion of generators of a pairwise comparisons matrix. Such approach decreases the number of pairwise comparisons from $n \cdot (n-1)$ to $n-1$. An algorithm of reconstructing of the PC matrix from its set of generators is presented.

1. Introduction

In [17], Thurstone proposed "The Law of Comparative Judgments" for pairwise comparisons (for short, PC). However, the first use of pairwise comparisons is in [5]). Even earlier use of PC is published in [3], but in a more simplified way for voting (win or loss). The PC theory finds a lot of applications, for example, in transport. An integrated simulation, multivariate analysis and multiple decision analysis for railway system improvement and optimization is presented in [1] where data envelopment analysis is used to solve the multi-objective model to identify the best alternatives.

In many cases, we meet the incomplete PC matrices which should be completed in such a way that they become consistent. The authors of [4] deal with this problem by means of similarity and parametric compromise functions. In [15], a fitness function is defined as a scalar vector function composed of the common error measure, based on the Euclidean distance, and a minimum violation error that accounts for no violation of the rank ordering is considered to improve deriving of the weights.

In this study, we examine the possibility of reconstructing the entire $n \times n$ PC matrix from only $n-1$ given entries placed in strategic locations. We call them PC-generators. Before we progress, some terminologies of pairwise comparisons must be revisited in the next section, since PC theory is still not as popular as other mathematical theories. However, the next section is definitely not for PC method experts.

2. Pairwise comparisons basics

We define an $n \times n$ *pairwise comparison matrix* simply as a square matrix $M = [m_{ij}]$ such that $m_{ij} > 0$ for every $i,j = 1,\dots,n$. A pairwise comparison matrix M is called *reciprocal* if $m_{ij} = \frac{1}{m_{ji}}$ for every $i,j = 1,\dots,n$ (then automatically $m_{ii} = 1$ for every $i = 1,\dots,n$). Let us assume that:

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$$M = \begin{bmatrix} 1 & m_{12} & \cdots & m_{1n} \\ \frac{1}{m_{12}} & 1 & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \cdots & 1 \end{bmatrix},$$

where m_{ij} expresses a relative quantity, intensity, or preference of entity (or stimuli) E_i over E_j. A more compact and elegant specification of PC matrix is given in [14] by Kulakowski.

A pairwise comparison matrix M is called *consistent* (or *transitive*) if:

$$m_{ij} \cdot m_{jk} = m_{ik},$$

for every $i, j, k = 1, 2, \ldots, n$.

We will refer to it as a "consistency condition". Consistent PC matrices correspond to the situation with the exact values $\mu(E_1), \ldots, \mu(E_n)$ for all the entities. In such case, the quotients $m_{ij} = \mu(E_i)/\mu(E_j)$ then form a consistent PC matrix. The vector $s = [\mu(E_1), \ldots \mu(E_n)]$ is unique up to a multiplicative constant. While every consistent matrix is reciprocal, the converse is generally false. If the consistency condition does not hold, the matrix is *inconsistent* (or *intransitive*). Axiomatization of inconsistency indicators for pairwise comparisons has been recently proposed in [13] and various inconsistency indexes are analyzed in [2].

The challenge for the pairwise comparisons method comes from the lack of consistency of the pairwise comparisons matrices, which arises in practice (while as a rule, all the pairwise comparisons matrices are reciprocal). Given an $N \times N$ matrix M, which is not consistent, the theory attempts to provide a consistent $n \times n$ matrix M', which differs from matrix M "as little as possible".

It is worth to note that the matrix: $M = [v_i/v_j]$ is consistent for all (even random) positive values v_i. It is an important observation since it implies that a problem of approximation is really a problem of a norm selection and the distance minimization. For the Euclidean norm, the vector of geometric means (equal to the principal eigenvector for the transitive matrix) is the one which generates it. Needless to say that only optimization methods can approximate the given matrix for the assumed norm (e.g., LSM for the Euclidean distance, as recently proposed in [7]). Such type of matrices are examined in [16] as "error-free" matrices.

It is unfortunate that the singular form "comparison" is sometimes used considering that a minimum of three comparisons are needed for the method to have a practical meaning. Comparing two entities (stimuli or properties) in pairs is irreducible, since having one entity compared with itself gives trivially 1. Comparing only two entities (2×2 PC matrix) does not involve inconsistency. Entities and/or their properties are often called stimuli in the PC research but are rarely used in applications.

3. The PC-generators of pairwise comparisons matrix

For a given PC matrix $A \in M_{n \times n}(\mathbb{R})$ consider the set $C_n := \{a_{ij} : \ i < j\}$. Note that in order to reconstruct the whole consistent matrix it is enough to know the elements of C_n, as $a_{ii} = 1$ for each $i \in \{1, \ldots, n\}$ and $a_{ji} = \frac{1}{a_{ij}}$ for $i < j$.

Let us call each such set sufficient to reconstruct the matrix A its *set of PC-generators*.

The set C_n has $\frac{n^2-n}{2}$ elements. However, consistency is a much stronger condition. So, it is obvious that we may reduce this input set by computing the rest of elements. It is a natural question to ask which minimal subsets of C_n generate A.

Remark 3.1. If $B \subset B' \subset C_n$ and B generates A, then B' does as well.

Theorem 3.2. *There is no $(n-2)$-set of PC-generators of A.*

Proof. For $n = 3$ the statement is obvious, as in any matrix:

$$\begin{bmatrix} 1 & a & c \\ \frac{1}{a} & 1 & b \\ \frac{1}{c} & \frac{1}{b} & 1 \end{bmatrix}.$$

if we only know one of the values a, b or c, we cannot clearly calculate the other two satisfying $c = ab$.

To continue the induction, let us assume that the assertion holds for each matrix $M \in M_{n \times n}(\mathbb{R})$. Now consider the matrix:

$$A_{n+1} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 & a_{n,n+1} \\ \frac{1}{a_{1,n+1}} & \frac{1}{a_{2,n+1}} & \cdots & \frac{1}{a_{n,n+1}} & 1 \end{bmatrix}.$$