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# Oscillation theorems for linear matrix Hamiltonian systems ☆

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ABSTRACT

Using a matrix linear transformation which preserves oscillatory property, some new oscillation criteria for the following linear matrix Hamiltonian system

$$U' = (A(t) - \lambda(t)I)U + B(t)V,$$
$$V' = C(t)U + (\mu(t)I - A^*(t))V, \quad t \geqslant t_0$$

are obtained, which improve and generalize some recent results in literature.



## 1. Introduction

Hamiltonian matrix differential systems arise in many dynamic problems and have been studied by many authors (e.g., see [1–19,30] and the references cited therein). In this paper, we consider oscillatory properties for the linear Hamiltonian matrix differential system

$$\begin{cases} U' = (A(t) - \lambda(t)I)U + B(t)V, \\ V' = C(t)U + (\mu(t)I - A^*(t))V, \end{cases} \tag{1.1}$$

where $A(t), B(t), C(t)$ are $n \times n$ matrices of real continuous matrix-valued functions on the interval $[t_0, \infty), B(t) = B^*(t), C(t) = C^*(t)$, here and in the sequel, the transpose of a matrix $M$ is denoted by $M^*$ and for any symmetric matrix $M$, by $M > 0$, it means $M$ is positive definite. $B(t)$ is either positive definite or negative definite, $\mu(t), \lambda(t)$ are real valued continuous function on $[t_0, \infty)$. When $\lambda(t) = \mu(t) \equiv 0$, system (1.1) reduces to the following Hamiltonian system

$$\begin{cases} U' = A(t)U + B(t)V, \\ V' = C(t)U - A^*(t)V. \end{cases} \tag{1.2}$$

A solution $(U(t), V(t))$ of the system (1.1) (or (1.2)) is said to be nontrivial if $\det U(t) \neq 0$ for at least one $t \in [t_0, \infty)$. A nontrivial solution $(U(t), V(t))$ of (1.1) (or (1.2)) is said to be prepared or self-conjugate if $U^*(t)V(t) - V^*(t)U(t) \equiv 0, t \geqslant t_0$. System (1.1) (or (1.2)) is said to be oscillatory on $[t_0, \infty)$ if there is a nontrivial prepared solution $(U(t), V(t))$ of (1.1) (or (1.2)) such that $\det U(t)$ vanishes at least once on $[T, \infty)$ for each $T \geqslant t_0$. Otherwise, it is said to be nonoscillatory.

When $A(t) \equiv 0$, system (1.2) reduces to the second order self-adjoint matrix differential system

$$(P(t)Y')' + Q(t)Y = 0, \quad t \geqslant t_0, \tag{1.3}$$

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with $P(t) = B^{-1}(t), Q(t) = -C(t)$. Oscillation and non-oscillation of system (1.3) and its special case

$$Y'' + Q(t)Y = 0, \quad t \geqslant t_0, \tag{1.4}$$

have been extensively studied by many authors (see [3,4,17,18,21,22,25–29] and references contained therein). Many of these criteria involve the integral of the coefficients modeled on either the criteria due to Wintner [23] or Kamenev [24] for the scalar equation. The Hamiltonian system (1.2) has also been investigated by many authors (see [5–20] for details). Most of these oscillation criteria involve the fundamental matrix $\Phi(t)$ for the linear system $v' = A(t)v$. Such a system cannot be solved easily when $A(t)$ is a continuous function of $t$. Moreover, by using the following transformation

$$\begin{pmatrix} \overline{U} \\ \overline{V} \end{pmatrix} = \begin{pmatrix} \Phi^{-1}(t) & 0 \\ 0 & \Phi^{*}(t) \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \tag{1.5}$$

we can transform (1.2) into the following Hamiltonian system

$$\begin{cases} \overline{U}' = \Phi^{*}(t)B(t)\Phi(t)\overline{V}, \\ \overline{V}' = \Phi^{-1}(t)C(t)\Phi^{*-1}(t)\overline{U}, \quad t \geqslant t_0. \end{cases} \tag{1.6}$$

We can easily rewrite (1.6) in the form (1.3) with $P(t) = \Phi^{-1}(t)B^{-1}(t)\Phi^{*-1}(t), Q(t) = -\Phi^{-1}(t)C(t)\Phi^{*-1}(t)$. So those criteria are similar to that of system (1.3). Oscillation criteria without the fundamental matrix $\Phi(t)$ of the linear system $v' = A(t)v$ for system (1.2) are fewer. By using a Kummer transformation, the author in [14] gave Kamenev type oscillation criterion for system (1.2) without the fundamental matrix $\Phi(t)$ as follows:

**Theorem 1.1.** Suppose that there exist $a(t) \in C^1([t_0, \infty); \mathbb{R}^+)$ and $f(t) = -\frac{a'(t)}{2a(t)}$ such that $fB^{-1}$ is differentiable. If there exists $H \in \mathcal{H}$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \lambda_1 \left\{ \int_{t_0}^{t} M(t,s)\mathrm{d}s \right\} = \infty,$$

where $\mathcal{H}$ is defined in Section 2 and

$$M(t,s) = H(t,s)D(s) - h(t,s)\sqrt{H(t,s)}K(s) - \frac{1}{4}h^2(t,s)\overline{B}^{-1}(s),$$

$$K(t) = \frac{1}{2}\left(A^{*}(\overline{B})^{-1} + (\overline{B})^{-1}A\right)(t), \quad D(t) = \left(-\overline{C} - A^{*}(\overline{B})^{-1}A\right)(t),$$

$$\overline{B}(t) = a^{-1}(t)B(t), \quad \overline{C}(t) = a(t)\left\{C + f(A^{*}B^{-1} + B^{-1}A) - f^2B^{-1} + (fB^{-1})'\right\}(t).$$

Then system (1.2) is oscillatory.

Moreover, paper [14] gave the oscillation criterion of Philos type for system (1.2) as follows.

**Theorem 1.2.** *Suppose that there exist $a(t) \in C^1([t_0, \infty); \mathbb{R}^+)$, and $f(t) = -\frac{a'(t)}{2a(t)}$ such that $fB^{-1}$ is differentiable. There exists $H \in \mathcal{H}$ such that*

$$0 < \inf_{s\geqslant t_0}\left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leqslant \infty, \tag{1.7}$$

$$\liminf_{t\to\infty} \tfrac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\mathrm{tr}D(s) - h(t,s)\sqrt{H(t,s)}\mathrm{tr}K(s)\right]ds > -\infty,$$
$$\limsup_{t\to\infty} \tfrac{1}{H(t,t_0)} \int_{t_0}^{t} \tfrac{a(s)h^2(t,s)}{\lambda_n(B(s))} ds < \infty.$$

Moreover, there exists another function $\psi(t) \in C([t_0, \infty); \mathbb{R})$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \lambda_1 \left\{ \int_{T}^{t} \left[H(t,s)D(s) - h(t,s)\sqrt{H(t,s)}K(s) - \frac{1}{4}a(s)h^2(t,s)B^{-1}(s)\right] ds \right\} \geqslant \psi(T)$$

for all $T \geqslant t_0$. Then system (1.2) is oscillatory provided

$$\int_{t_0}^{\infty} \frac{\lambda_n(B(s))}{a(s)} \left[\psi(s) + \lambda_n(\overline{K}(s))\right]_+^2 ds = \infty,$$

where $D(t)$ and $K(t)$ are the same as in Theorem 1.1.

Using positive linear functional on $\mathcal{M}$ (where $\mathcal{M}$ is defined in Section 2), Li et al. [5] obtained improved Philos type oscillation criteria for system (1.2).