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Symmetry solutions for reaction–diffusion equations with spatially dependent diffusivity

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ABSTRACT

Nonclassical and classical symmetry techniques are employed to analyse a reaction–diffusion equation with a cubic source term. Here, the diffusivity (diffusion term) is assumed to be an arbitrary function of the spatial variable. Classification using Lie point and nonclassical symmetries is performed. It turns out that the diffusivity needs to be given as a quadratic function of the spatial variable for the given governing equation to admit nonclassical symmetries. Both nonclassical and classical symmetries are used to construct some group-invariant (exact) solutions. The results are applied to models arising in population dynamics.

1. Introduction

Classical Lie point and nonclassical symmetry analysis are useful techniques when searching for analytic solutions to differential equations. The main difference underlying the determination of classical Lie point and nonclassical symmetries (also known as conditional symmetries, Q-conditional symmetries or reduction operators [1]) is in the condition imposed in the infinitesimal criterion for invariance. For classical symmetries it is required that the differential equation in question be invariant on solutions of the equation itself, however for the determination of nonclassical symmetries, an extra condition is that the equation is invariant both on its solutions and its invariant surface condition (see for example [2]). Bluman and Cole [2] pioneered the idea of nonclassical symmetries. Nonclassical symmetry methods may give rise to exact solutions which cannot be obtained by classical methods (see for example [3]).

Nonclassical symmetry methods have been applied to reaction–diffusion equations or the nonlinear diffusion equation with a source term (among others see e.g. [4–12]). Serov [4], and later Arrigo et al. [5] and Clarkson and Mansfield [6], found that the quasilinear reaction–diffusion equation

$$u_t = u_{xx} + Q(u),$$

with constant diffusivity rescaled to one, admits strictly nonclassical symmetries only if $Q(u)$ is a cubic polynomial. Nucci and Clarkson [13] also found solutions for particular forms of cubic $Q(u)$. Vaneeva and Zhalij [14] performed the group classification via equivalence transformations of the reaction–diffusion (generalised Huxley) equation

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$$u_t = u_{xx} + g(x)Q(u),$$

while the nonclassical symmetry analysis of this equation with $Q(u)$ cubic was carried out in [8] and with $Q(u) = u^2(1-u)$ in [15]. Hashemi and Nucci [9] applied nonclassical symmetry analysis to a related class of reaction–diffusion equations.

In this paper, we use assume that the diffusivity depends on the space variable so that the equation in question is given by

$$F = u_t - (k(x)u_x)_x - Q(u) = 0, \quad (1)$$

where $Q(u)$ is a factorisable cubic. We refer to Eq. (1) as the governing equation. Conditions under which a general PDE may be mapped to another are discussed in [16]. As mentioned above, for Eq. (1), in the case where $k(x) = k_0 \equiv \text{constant} \neq 0$ and $Q(u)$ is cubic, nonclassical solutions have been found [4–6,10,13]. Here, we employ the classical and nonclassical symmetry techniques to construct exact (group-invariant) solutions for Eq. (1) when $k(x) \neq$ constant. Equations of this type (1) can be used to model problems arising in heat conduction [17], biology (population dynamics [7,18,19], transmission of nerve signals [20]), and combustion theory [21]. Exact solutions play an important role in identifying interesting behaviours in nonlinear systems.

This paper is organised as follows; in Section 2 we discuss the classical and nonclassical symmetry techniques in brief. In Section 3, we provide analysis of the governing equation using nonclassical symmetry techniques to construct some exact (group-invariant) solutions. In Section 4, solutions are found using classical Lie point symmetries. In Section 5 we discuss a possible application of the constructed solutions to a problem in population dynamics. Lastly, we provide some final remarks in Section 6.

2. Classical and nonclassical symmetry analysis

In brief, a Lie point symmetry of a differential equation is an invertible transformation of the independent and dependent variables that preserves the governing equation and depends smoothly on a continuous parameter. The literature in this area is sizable (see for example [22–26]) and will only be described in brief here. To find symmetries of a second order partial differential equation,

$$\Delta = G(t, x, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}) = 0,$$

one seeks a one-parameter group of transformations generated by the base vector

$$\Gamma = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u},$$

which leave the governing equation invariant. The operator Γ is a classical Lie point symmetry of the governing equation provided the invariance criteria

$$\Gamma^{[2]}(\Delta)|_{\Delta=0} = 0 \quad (2)$$

holds, where $\Gamma^{[2]}$ is the second prolongation of Γ (defined in [25], for example).

Eq. (2) results in an overdetermined system of *linear* homogeneous partial differential equations, known as the determining equations. Albeit tedious, these equations can be algorithmically solved. Calculations can be done by hand or facilitated by interactive computer software such as Maple [27] or Reduce [28].

The nonclassical symmetry techniques developed by Bluman and Cole [2] generalise Lie's classical symmetry method for obtaining analytic solutions to partial differential equations. We seek invariance of the governing equation subject to the constraint of the invariant surface condition (ISC)

$$\xi^1 u_t + \xi^2 u_x = \eta,$$

which can sometimes lead to additional reductions which are not obtainable using the classical method.

In terms of the second prolongation of the nonclassical symmetry generator, $\Gamma^{[2]}$, the determining relations for parabolic PDEs are given by

$$\Gamma^{[2]}(\Delta)|_{\Delta=0,\ \text{ISC}} = 0. \quad (3)$$

Unlike in the case of searching for classical Lie point symmetries, the determining equations resulting from (3) are a set of overdetermined *nonlinear* equations which must be solved to find $\xi^1(x,t,u), \xi^2(x,t,u)$ and $\eta(x,t,u)$. For 1+1 dimensional evolution equations, the case where $\xi^1 = 0$ is known to be unfeasible since finding any nonclassical operators is equivalent to solving the original equation [29]. Therefore, we restrict our study to the case of regular nonclassical operators and without loss of generality set $\xi^1 = 1$ (see also [3–6]). Solution of the determining equations may require specific restrictions to be placed on arbitrary functions – in the case of Eq. (1), $k(x)$. Again, calculations may be facilitated by an interactive code in Maple.