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A quadratic convergence yielding iterative method for the implementation of Lavrentiev regularization method for ill-posed equations

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ABSTRACT

George and Elmahdy (2012), considered an iterative method which converges quadratically to the unique solution x_α^δ of the method of Lavrentiev regularization, i.e., $F(x) + \alpha(x - x_0) = y^\delta$, approximating the solution $\hat{x}$ of the ill-posed problem $F(x) = y$ where $F : D(F) \subseteq X \longrightarrow X$ is a nonlinear monotone operator defined on a real Hilbert space X. The convergence analysis of the method was based on a majorizing sequence. In this paper we are concerned with the problem of expanding the applicability of the method considered by George and Elmahdy (2012) by weakening the restrictive conditions imposed on the radius of the convergence ball and also by weakening the popular Lipschitz-type hypotheses considered in earlier studies such as George and Elmahdy (2012), Mahale and Nair (2009), Mathe and Perverzev (2003), Nair and Ravishankar (2008), Semenova (2010) and Tautanhahn (2002). We show that the adaptive scheme considered by Perverzev and Schock (2005) for choosing the regularization parameter can be effectively used here for obtaining order optimal error estimate. In the concluding section the method is applied to numerical solution of the inverse gravimetry problem.

1. Introduction

Let X be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let $B_r(x)$ and $\overline{B_r(x)}$, stand respectively, for the open and closed ball in X with center $x \in X$ and radius $r > 0$. In this paper, we are interested in obtaining a stable approximate solution for a nonlinear ill-posed operator equation of the form

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator. Note that F is a monotone operator if it satisfies the relation $\langle F(u) - F(v),\ u - v \rangle \geqslant 0$ for all $u,\ v \in D(F)$.

It is assumed that (1.1) has a solution, say $\hat{x}$, and F is Fréchet differentiable for all $x \in D(F)$. Further, we assume that $y^\delta \in X$ are the available noisy data with

$$\|y - y^\delta\| \leqslant \delta. \tag{1.2}$$

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Since (1.1) is ill-posed, its solution need not depend continuously on the data, i.e., small perturbation in the data can cause large deviation in the solution. So, one has to use regularization method [3–7,9–21]. Since F is monotone, one may use the Lavrentiev regularization method [6,7,9]. In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{1.3}$$

It is known (cf. [21, Theorem 1.1]) that (1.3) has unique solution x_α^δ for $\alpha \geqslant 0$, provided F is Fréchet differentiable and monotone in the ball $B_r(\hat{x}) \subset D(F)$ with radius $r = \|\hat{x} - x_0\| + \frac{\delta}{\alpha}$ (in Section 2 we prove that (1.3) has a unique solution for all $x \in B_r(x_0)$ under some assumption on the Fréchet derivative of F). However the regularized Eq. (1.3) remains nonlinear and one may have difficulties in solving it numerically.

In [8], George and Elmahdy considered the method defined iteratively by

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - (F'(x_{n,\alpha}^\delta) + \alpha I)^{-1}\left[F(x_{n,\alpha}^\delta) - y^\delta + \alpha(x_{n,\alpha}^\delta - x_0)\right], \tag{1.4}$$

where $x_0 := x_{0,\alpha}^\delta$ is the starting point of the iteration for approximately solving (1.3). They proved that $x_{n,\alpha}^\delta$ converges quadratically to x_α^δ. The convergence analysis in [8], was based on a majorizing sequence and the conditions (see (2.10) and (2.11) in [8]) required for the convergence of the method are not easy to verify. This is the main drawback of the analysis in [8]. The convergence analysis in [8] was carried out using the following assumptions.

Assumption 1.1. There exists $r > 0$ such that $B_r(x_0) \cup B_r(\hat{x}) \subset D(F)$ and F is Fréchet differentiable at all $x \in B_r(x_0) \cup B_r(\hat{x})$.

Assumption 1.2. There exists a constant $K_0 > 0$ such that for every $u,\ v \in B_r(x_0) \cup B_r(\hat{x})$ and $w \in X$, there exists an element $\phi(u, v, w) \in X$ satisfying $[F'(u) - F'(v)]w = F'(v)\phi(u, v, w), \|\phi(u, v, w)\| \leqslant K_0\|w\|\|u - v\|$.

Assumption 1.3. There exists a continuous and strictly increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geqslant \|F'(\hat{x})\|$ satisfying;

(i) $lim_{\lambda \to 0}\varphi(\lambda) = 0$,
(ii) $sup_{\lambda \geqslant 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leqslant c_\varphi\varphi(\alpha) \quad \forall \lambda \in (0, a]$ and
(iii) there exists $v \in X$ with $\|v\| \leqslant 1$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v.$$

In the present paper, we replace Assumption 1.3 by

Assumption 1.4. There exists a continuous and strictly increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geqslant \|F'(x_0)\|$ satisfying;

(i) $lim_{\lambda \to 0}\varphi(\lambda) = 0$,
(ii) $sup_{\lambda \geqslant 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leqslant \varphi(\alpha) \quad \forall \lambda \in (0, a]$ and
(iii) there exists $v \in X$ with $\|v\| \leqslant 1$ such that

$$x_0 - \hat{x} = \varphi(F'(x_0))v.$$

In Section 3 we replace Assumption 1.2 by

Assumption 1.5. Suppose there exists a constant $K_0 > 0$ such that for all $w \in X$ and $u,\ v \in B_r(x_0) \subseteq D(F)$, there exists element $\phi(u, v, w) \in X$ such that $[F'(u) - F'(v)]w = F'(v)\phi(u, v, w)$ and $\|\phi(u, v, w)\| \leqslant K_0\|w\|\|u - v\|$.

Remark 1.6. If Assumption 1.5 is fulfilled only for all $u,\ v \in B_r(x_0) \cap Q \neq \emptyset$, where Q is a convex closed a priori set for which $\hat{x} \in Q$, then we can modify the method (1.4) in the following way:

$$x_{n+1,\alpha}^\delta = P_Q(T(x_{n,\alpha}^\delta))$$

to obtain the same estimate in Theorem 3.1. Here P_Q is the metric projection onto the set Q and T is the step operator in the method (1.4).

Using the above assumption, we prove that the method (1.4) converges quadratically to the solution x_α^δ of (1.3).

Recall that, a sequence (x_n) is said to be converging quadratically to x^*, if there exists a positive number M_q, not necessarily less than 1, such that

$$\|x_{n+1} - x^*\| \leqslant M_q\|x_n - x^*\|^2,$$

for all n sufficiently large. And the convergence of (x_n) to x^*, is said to be linear if there exists a positive number $M_0 \in (0, 1)$, such that