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Beta-type polynomials and their generating functions

Yilmaz Simsek

Department of Mathematics, Faculty of Science, University of Akdeniz, TR-07058 Antalya, Turkey

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ABSTRACT

We construct generating functions for beta-type rational functions and the beta polynomials. By using these generating functions, we derive a collection of functional equations and PDEs. By using these functional equations and PDEs, we give derivative formulas, a recurrence relation and a variety of identities related to these polynomials. We also give a relation between the beta-type rational functions and the Bernstein basis functions. Integrating these identities and relations, we derive various combinatorial sums involving binomial coefficients, some old and some new, for the beta-type rational functions and the Bernstein basis functions. Finally, by applying the Laplace transform to these generating functions, we obtain two series representations for the beta-type rational functions.

1. Introduction

Polynomials are often used in both mathematics and engineering applications because they are closed under addition, multiplication, differentiation, integration and composition. Therefore, they have become an efficient tool to represent different systems, simulate various processes and compute final decisions. Recently, generating functions for special polynomials and numbers have also been used in many branches of mathematics, physics and engineering. Using generating functions, many fundamental properties and identities for special polynomials and numbers can be obtained.

In this paper, we study generating functions for the beta polynomials and the beta-type rational functions. In [4], Bhandari and Vignat studied the beta polynomials without generating functions. By using these polynomials, they gave a probabilistic representation of the multidimensional Volkenborn integral. They also obtained the expectation of the beta polynomials. Thus these polynomials have also been used in the theory of probability distributions. These polynomials are very important to compute combinatorial sums involving binomial coefficients.

We define new function, which are called beta-type rational functions:

Definition 1.1. Let n and k be nonnegative integers. The beta-type rational functions $\mathfrak{M}_{k,n}(x)$ are defined by the following formula:

$$\mathfrak{M}_{k,n}(x) = x^k(1+x)^{n-k}. \tag{1}$$

The beta polynomials $\mathfrak{B}_{k,n}(x)$ are defined as follows:

E-mail address: ysimsek@akdeniz.edu.tr

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Definition 1.2. Let n and k be nonnegative integers. The beta polynomials of degree n on the interval $[-1, 0]$ are defined by

$$\mathfrak{B}_{k,n}(x) = x^k(1+x)^{n-k}. \tag{2}$$

where $k = 0, 1, 2, \ldots, n$.

By using the Volkenborn integral and probability distribution, Bhandari and Vignat [4] studied the beta polynomials. In [13], by using generating functions, we studied these polynomials. We also gave some identities and relations related to these polynomials. In order to investigate some properties of the beta-type rational functions and also the beta polynomials, we construct generating functions to derive some known and many new identities for these polynomials. By applying integrals to these generating functions and these identities, we derive many combinatorial sums involving binomial coefficients. In many branches of mathematics, such as number theory, combinatorial analysis, graph theory, binomial coefficients appear. Therefore, combinatorial sums involving binomial coefficients play an important role in all branches of Mathematics. By applying the Laplace transform to these generating functions, two function series representations for the beta-type rational functions are given.

We summarize our results as follows:

In Section 2, we give generating functions for the beta-type polynomials and the beta polynomials. We investigate some properties of these generating functions. We also give some functional equations for these generating functions. In Section 3, by using functional equations of the generating functions, we derive many identities related to the beta-type rational functions and the beta polynomials. We give a relation between the Beta polynomials and the Bernstein basis functions. In Section 4, we give PDEs for the generating functions. Using these PDEs, we derive two derivative formulas and a recurrence formula for the beta-type rational functions. In Section 5, by using the gamma function and the beta function, we give formulas that are associated with integrals of the beta polynomials. Using these formulas and identities related to the beta-type rational functions, we compute combinatorial identities and combinatorial sums involving binomial and inverse binomial coefficients. We also give a numerical example related to combinatorial sums. In Section 6, by using the Laplace transform, we find two series representations for the beta-type rational functions.

2. Generating functions

In this section, we construct generating functions for the beta-type rational functions and the beta polynomials. We investigate some properties of these generating functions. In order to give derivative formulas, a recurrence relation and a variety of identities related to the beta-type rational functions and the beta polynomials, we derive many functional equations and PDEs for these generating functions. Some of these functional equations and PDEs are given in the next sections.

The beta-type rational functions $\mathfrak{M}_{k,n}(x)$ are defined by means of the following generating functions:

$$\mathfrak{h}_k(t,x) = \left(\frac{x}{1+x}\right)^k e^{t(1+x)} = \sum_{n=0}^{\infty} \mathfrak{M}_{k,n}(x)\frac{t^n}{n!}, \tag{3}$$

where k is a nonnegative integer and $t \in \mathbb{C}$, set of complex numbers.

There is one generating function for each value of k.

By using (3), we also construct generating functions for the beta polynomials as follows:

$$\mathfrak{F}_k(t,x) = \sum_{n=k}^{\infty} \mathfrak{B}_{k,n}(x)\frac{t^n}{n!},$$

where

$$\mathfrak{F}_k(t,x) = \mathfrak{h}_k(t,x) - \sum_{n=0}^{k-1} \frac{x^k}{(1+x)^{k-n}}\frac{t^n}{n!}.$$

2.1. Some properties of the function $\mathfrak{h}_k(t,x)$

We introduce and investigate some properties of the generating functions $\mathfrak{h}_k(t,x)$ and we derive some functional equations for these generating functions. By using these generating functions, we also construct generating functions for the beta polynomials.

Theorem 2.1. *Let* $\left|\frac{x}{1+x}\right| < 1$. *Then we have*

$$\frac{d}{dt}e^{t(1+x)} = \sum_{k=0}^{\infty} \mathfrak{h}_k(t,x). \tag{4}$$