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On the real-rootedness of generalized Touchard polynomials

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ABSTRACT

We consider the real-rootedness of generalized Touchard polynomials recently revisited by Mansour and Schork (2013). Towards this end, we first describe the normal form of the generalized Touchard polynomials, by which recurrence relations for the polynomial part are derived. By using the recurrence relations, we prove the real-rootedness of the generalized Touchard polynomials for the parameter $m \in [1, \infty) \cup \left\{\frac{k}{k+1} : k \in \mathbb{N}\right\}$.

1. Introduction

Jacques Touchard [17,18] introduced in 1933 the Touchard polynomials $T_n(x)$'s, in the study of permutations with cycles satisfying certain conditions, as extensions of the partial Bell polynomials [1]. The Touchard polynomials $T_n(x)$'s (also known as the *exponential polynomial* or *Bell polynomial*) can either be defined by the exponential generating function $\sum_{n\geqslant 0} T_n(x)\frac{t^n}{n!} = e^{x(e^t-1)}$ or by the Stirling transform $T_n(x) = \sum_{j=0}^{n} S_{n,j}x^j$, where $S_{n,j} = \frac{1}{j!}\sum_{\ell=0}^{j}(-1)^{j-\ell}\binom{j}{\ell}\ell^n$ denotes the Stirling number of the second kind, which counts the number of partitions of an n-set into j non-empty blocks (see [12]).

It is well known that the moments of the Poisson distribution are intimately related to the combinatorics of Stirling numbers of the second kind and Bell numbers. More precisely, if the random variable $X \sim \text{Poisson}(\lambda)$, then for $n = 1, 2, \ldots$, the Touchard polynomial $T_n(x)$ satisfies the relation

$$T_n(\lambda) = E[X^n],$$

i.e., $T_n(\lambda)$ is the nth moment of X. Moreover, several properties of Touchard polynomials are studied in [3,5,10,18], where some of them are applied to problems in random walks.

Touchard polynomials have been extended in many contexts, see [4,5,10,14–16] for instance. Of relevance to the present work is the higher order extension of Touchard polynomials introduced by Dattoli et al. [7], defined for $m \in \mathbb{Z}, n \in \mathbb{N}$ by

$$T_n^{(m)}(x) := e^{-x}(x^m\partial_x)^n e^x, \tag{1.1}$$

which reduce when $m = 1$ to the classical Touchard polynomials $T_n(x)$ mentioned above. Dattoli et al. [7] discussed several properties of these polynomials, including the recurrence (1.2) below for $m \in \mathbb{N}$ (also, see [11]). Mansour and Schork [13] recently revisited $T_n^{(m)}(x)$ by extending m to arbitrary real number and showed various properties of $T_n^{(m)}(x)$'s, including the following recurrence relation

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$$(x^m + x^m\partial_x)T_n^{(m)}(x) = T_{n+1}^{(m)}(x) \tag{1.2}$$

as well as an identity relating $T_n^{(\frac{1}{2})}(x)$ and the nth Hermite polynomial $H_n(x)$. See (4.1) in Section 4. Since $H_n(x)$ possesses the remarkable property of being real-rooted, this suggests that $T_n^{(\frac{1}{2})}(x)$ enjoys this remarkable property also. A crucial property of $T_n^{(\frac{1}{2})}(x)$ not explored by Mansour and Schork is the representation $T_n^{(\frac{1}{2})}(x) = x^{\alpha_n}\tilde{T}_n^{(\frac{1}{2})}(x)$, where $\alpha_n \in \{0, \frac{1}{2}\}$ and $\tilde{T}_n^{(\frac{1}{2})} \in \mathbb{Q}^+[x]$. Since the real-rootedness of $T_n^{(\frac{1}{2})}(x)$ implies that of $\tilde{T}_n^{(\frac{1}{2})}(x)$, an application of the Aissen–Schoenberg–Whitney theorem [2, Theorem 2.2.4] to $\tilde{T}_n^{(\frac{1}{2})}(x) = \sum_{i=0}^d a_i x^i$ then yields the total positivity consequence that any minor of the infinite matrix $M = (M_{ij})_{i,j\in\mathbb{N}}$ defined by $M_{ij} = a_{j-i}$ for all $i,j \in \mathbb{N}$ (where $a_k = 0$ if $k < 0$ or $k > d = \deg \tilde{T}_n^{(\frac{1}{2})}(x)$) is non-negative. Because of these, it is of considerable interest to study more generally the real-rootedness of $T_n^{(m)}(x)$ for $m \in \mathbb{R}$. The organization of this paper is as follows. In the next section, we consider the positive integral m case. In Section 3, we consider the negative integral m case. In Section 4, we study how the positive integral m case extends to give the positive real m case.

2. The positive integer case

We study in the present section the real-rootedness of $T_n^{(m)}(x)$ for $m \in \mathbb{N}$.

Let $m \in \mathbb{N}$. One can compute $T_n^{(m)}(x)$ either directly by (1.1), or by the recurrence (1.2). The first few members are listed as follows:

$$\begin{aligned}
T_1^{(m)}(x) &= x^m,\\
T_2^{(m)}(x) &= x^{2m-1}(x+m),\\
T_3^{(m)}(x) &= x^{3m-2}[x^2 + 3mx + m(2m-1)],\\
T_4^{(m)}(x) &= x^{4m-3}[x^3 + 6mx^2 + m(11m-4)x + m(2m-1)(3m-2)],\\
T_5^{(m)}(x) &= x^{5m-4}[x^4 + 10mx^3 + 5m(7m-2)x^2 + 5m(2m-1)(5m-2)x + m(2m-1)(3m-2)(4m-3)].
\end{aligned}$$

We see that when $m \in \mathbb{N}$, all coefficients of $T_n^{(m)}(x)$ are positive integral.

Proposition 1. *For any positive integers m and k, $T_k^{(m)}(x)$ is a monic polynomial of degree km with non-negative integer coefficients.*

Proof. We fix the positive integer m and proceed by induction on k. Since

$$T_1^{(m)}(x) = e^{-x}(x^m\partial_x)e^x = x^m,$$

the theorem clearly holds when $k = 1$. Assume that the result holds up to k. By the recurrence relation

$$T_{k+1}^{(m)}(x) = (x^m + x^m\partial_x)T_k^{(m)}(x) = x^mT_k^{(m)}(x) + x^m(T_k^{(m)})'(x),$$

which is a monic polynomial of degree $m + km = (k+1)m$. □

It is evident from the above list that

$$T_n^{(m)}(x) = x^{nm-n+1}\tilde{T}_n^{(m)}(x) \tag{2.1}$$

for some monic polynomial $\tilde{T}_n^{(m)} \in \mathbb{N}[x]$ with $\tilde{T}_n^{(m)}(0) > 0$. We shall show that a sufficient condition for the coefficients of $\tilde{T}_n^{(m)}(x)$ to be positive integral for all n is that $m \in \mathbb{N}$.

Proposition 2. *For $n \in \mathbb{N}$, the Touchard polynomial $\tilde{T}_n^{(m)}(x)$ is monic, and satisfies the recurrence relation*

$$\tilde{T}_{n+1}^{(m)}(x) = (x + n(m-1) + 1)\tilde{T}_n^{(m)}(x) + x\left(\tilde{T}_n^{(m)}\right)'(x). \tag{2.2}$$

If $m \in \mathbb{N}$, then $\tilde{T}_n^{(m)} \in \mathbb{N}[x]$.

Proof. Substituting (2.1) into (1.2), we have

$$x^{(n+1)m-n}\tilde{T}_{n+1}^{(m)}(x) = (x^m + x^m\partial_x)x^{nm-n+1}\tilde{T}_n^{(m)}(x) = x^{(n+1)m-n}[(x + nm - n + 1)\tilde{T}_n^{(m)}(x) + x(\tilde{T}_n^{(m)})'(x)].$$

Cancelling $x^{(n+1)m-n}$ from both sides, (2.2) follows. The multiplicative factor $x + nm - n + 1$ has positive coefficients $\iff m > 1 - \frac{1}{n}$. Thus, for $m \in \mathbb{N}$, the latter condition holds and both factors on the right side of (2.2) have positive integral coefficients; $\tilde{T}_n^{(m)} \in \mathbb{N}[x]$ then follows by induction. □