Applied Mathematics and Computation 254 (2015) 210–218

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the numerical solutions of high order stable difference schemes for the hyperbolic multipoint nonlocal boundary value problems

Ozgur Yildirim *, Meltem Uzun

Department of Mathematics, Yildiz Technical University, Turkey

ARTICLE INFO

Keywords:
Abstract hyperbolic equations
Stability
Difference equations
Nonlocal and multipoint BVPs

ABSTRACT

In this article, we consider third and fourth order of accuracy stable difference schemes for the approximate solutions of hyperbolic multipoint nonlocal boundary value problem in a Hilbert space H with self-adjoint positive definite operator A. We present stability estimates and numerical analysis for the solutions of the difference schemes using finite difference method.

1. Introduction

Hyperbolic partial differential equations (PDEs) with nonlocal boundary conditions play an important role in several areas such as engineering and natural sciences, in particular acoustic, electromagnetic, hydrodynamic, elasticity, fluid mechanics, and other areas of physics (see, e.g., [1–5] and the references given therein). In the development of numerical methods stability has been studied by many scientists (see [5–28] and the references given therein).

In this paper, third and fourth order of accuracy difference schemes for the approximate solution of multipoint nonlocal boundary-value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m} (a_r(x)u_{x_r})_{x_r} = f(t,x), \\ x = (x_1,\ldots,x_m) \in \Omega, \quad 0 < t < 1, \\ u(0,x) = \sum_{j=1}^{n} \alpha_j u(\lambda_j, x) + \varphi(x), x \in \bar{\Omega}, \\ u_t(0,x) = \sum_{j=1}^{n} \beta_j u_t(\lambda_j, x) + \psi(x), x \in \bar{\Omega}, \\ u(t,x) = 0, \quad x \in S \end{cases} \tag{1}$$

for the multidimensional hyperbolic equation with Dirichlet condition is considered. Here Ω be the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m\{x = (x_1,\ldots,x_m) : 0 < x_j < 1,\ 1 \leqslant j \leqslant m\}$ with boundary S, $\bar{\Omega} = \Omega \cup S$. The stability estimates for solutions of the difference schemes are presented. Some results of numerical experiments supporting our theoretical statements are obtained.

* Corresponding author.
E-mail addresses: ozgury@yildiz.edu.tr (O. Yildirim), meltemu@yildiz.edu.tr (M. Uzun).

http://dx.doi.org/10.1016/j.amc.2014.12.117
0096-3003/

Note that many scientists have been studied on the solutions of boundary value problems such as parabolic equations, elliptic equations and equations of mixed types extensively (see, e.g., [17–28] and the references therein).

2. Stability estimates

In this section we will present stability estimates for the solutions of third and fourth order of accuracy difference schemes. In the first step, let us define the grid sets

$$\tilde{\Omega}_h = \{x = x_r = (h_1 r_1, \ldots, h_m r_m), \quad r = (r_1, \ldots, r_m), \quad 0 \leqslant r_j \leqslant N_j, h_j N_j = 1, j = 1, \ldots, m\}, \quad \Omega_h = \tilde{\Omega}_h \cap \Omega, S_h = \tilde{\Omega}_h \cap S.$$

We introduce the Banach space $L_{2h} = L_2(\tilde{\Omega}_h)$, $W^1_{2h} = W^1_{2h}\left(\tilde{\Omega}_h\right)$ and $W^2_{2h} = W^2_{2h}\left(\tilde{\Omega}_h\right)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 r_1, \ldots, h_m r_m)\}$ defined on $\tilde{\Omega}_h$, equipped with norms

$$\|\varphi^h\|_{L_2(\tilde{\Omega}_h)} = \left(\sum_{x\in\tilde{\Omega}_h} |\varphi^h(x)|^2 h_1 \ldots h_m\right)^{1/2},$$

$$\|\varphi^h\|_{W^1_{2h}} = \|\varphi^h\|_{L_{2h}} + \left(\sum_{x\in\tilde{\Omega}_h}\sum_{r=1}^{m}\left|(\varphi^h)_{\bar{x}_r,j_r}\right|^2 h_1 \ldots h_m\right)^{1/2}$$

and

$$\|\varphi^h\|_{W^2_{2h}} = \|\varphi^h\|_{L_{2h}} + \left(\sum_{x\in\tilde{\Omega}_h}\sum_{r=1}^{m}\left|(\varphi^h)_{\bar{x}_r}\right|^2 h_1 \ldots h_m\right)^{1/2} + \left(\sum_{x\in\tilde{\Omega}_h}\sum_{r=1}^{m}\left|(\varphi^h)_{x_r\bar{x}_r,j_r}\right|^2 h_1 \ldots h_m\right)^{1/2},$$

respectively. The difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m}\left(a_r(x) u^h_{\bar{x}_r}\right)_{x_r,j_r} \tag{2}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$ is considered. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\tilde{\Omega}_h)$. With the help of A_h^x we arrive at the nonlocal boundary value problem

$$\begin{cases} \frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), \quad 0 < t < 1, \quad x \in \Omega_h, \\ v^h(0,x) = \sum_{j=1}^{n} \alpha_j v^h(\lambda_j, x) + \varphi^h(x), \quad x \in \tilde{\Omega}_h, \\ \frac{dv^h(0,x)}{dt} = \sum_{j=1}^{n} \beta_j v^h_t(\lambda_j, x) + \psi^h(x), \quad x \in \tilde{\Omega}_h \end{cases} \tag{3}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3) by the following third order of accuracy difference scheme

$$\begin{cases} \tau^{-2}\left(u^h_{k+1}(x) - 2u^h_k(x) + u^h_{k-1}(x)\right) + \frac{2}{3}A_h^x u_k^h(x) + \frac{1}{6}A_h^x\left(u^h_{k+1}(x) + u^h_{k-1}(x)\right) + \frac{1}{12}\tau^2\left(A_h^x\right)^2 u^h_{k+1}(x) = f_k^h(x), \ f_k^h(x) = \frac{2}{3}f^h(t_k, x) \\ \quad + \frac{1}{6}\left(f^h(t_{k+1}, x) + f^h(t_{k-1}, x)\right) - \frac{1}{12}\tau^2\left(-Af^h(t_{k+1}, x) + f^h_{tt}(t_{k+1}, x)\right), \\ t_k = k\tau, \ 1 \leqslant k \leqslant N-1, \ N\tau = 1, \ x \in \Omega_h, \\ u_0^h(x) = \sum_{j=1}^{n}\alpha_j\left\{u^h_{[\lambda_j/\tau]}(x) + \tau^{-1}\left(u^h_{[\lambda_j/\tau]}(x) - u^h_{[\lambda_j/\tau]-1}(x)\right)(\lambda_j - [\lambda_j/\tau]\tau) + \frac{3}{2}\left(f_{[\lambda_j/\tau]} - A_h^x u^h_{[\lambda_j/\tau]}(x)\right)(\lambda_j - [\lambda_j/\tau]\tau)^2\right. \\ \quad \left. + \frac{7}{6}\left(f'_{[\lambda_j/\tau]} - \tau^{-1}A_h^x\left(u^h_{[\lambda_j/\tau]}(x) - u^h_{[\lambda_j/\tau]-1}(x)\right)\right)(\lambda_j - [\lambda_j/\tau]\tau)^3\right\} + \varphi^h(x), x \in \Omega_h, \\ \left(I + \tau^2\left(A_h^x\right)^4\right)\tau^{-1}\left(u_1^h(x) - u_0^h(x)\right) = \sum_{j=1}^{n}\beta_j\left\{\tau^{-1}\left(u^h_{[\lambda_j/\tau]}(x) - u^h_{[\lambda_j/\tau]-1}(x)\right) + \left(f_{[\lambda_j/\tau]} - A_h^x u^h_{[\lambda_j/\tau]}(x)\right)(\lambda_j - [\lambda_j/\tau]\tau)\right. \\ \quad \left. + \frac{1}{2!}\left(f'_{[\lambda_j/\tau]} - \tau^{-1}A_h^x\left(u^h_{[\lambda_j/\tau]}(x) - u^h_{[\lambda_j/\tau]-1}(x)\right)\right)(\lambda_j - [\lambda_j/\tau]\tau)^2 + \frac{1}{3!}\left(f''_{[\lambda_j/\tau]} - A_h^x f_{[\lambda_j/\tau]} + \left(A_h^x\right)^2 u^h_{[\lambda_j/\tau]}(x)\right)(\lambda_j - [\lambda_j/\tau]\tau)^3\right\} + \psi^h(x), x \in \Omega_h, \\ f^h_{1,1}(x) = \frac{1}{2}f^h(0,x) + \frac{\tau}{6}f^h_t(0,x). \end{cases}$$