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A stable family with high order of convergence for solving nonlinear equations ☆

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ABSTRACT

Recently, Li et al. (2014) have published a new family of iterative methods, without memory, with order of convergence five or six, which are not optimal in the sense of Kung and Traub's conjecture. Therefore, we attempt to modify this suggested family in such a way that it becomes optimal. To this end, we consider the same two first steps of the mentioned family, and furthermore, we introduce a better approximation for $f'(z)$ in the third step based on interpolation idea as opposed to the Taylor's series used in the work of Li et al. Theoretical, dynamical and numerical aspects of the new family are described and investigated in details.

1. Introduction

Iterative methods for approximating simple zeros of a real-valued function is an active research area which has progressed thanks to the advances in modern computers both in software and hardware. The principal base for constructing these methods is the significant and substantial works by Traub's [2] and Kung and Traub's [3]. In other words, Traub classified iterative methods in some sense. Here, we are interested in iterative multi-point methods without memory. Regarding the construction of these methods, we need and recall two basic criteria. Traub says for constructing a one-point method, having convergence order p, we require p functional evaluations. Designs and developments of one-point methods is not a considerable task because of its less efficiency. On the other hand, Kung and Traub conjectured that any multi-point iterative method, without memory, with $d+1$ functional evaluations per step, has an order of convergence at most 2^d. When this bound is reached the method is called *optimal*. Indeed, these methods overcome theoretical and practical issue of single point methods regarding computational evaluations and convergence rate.

As far as we know, Ostrowski's [4], Jarratt's [5], and King's [6] methods are the first optimal two-point methods of fourth-order. Moreover, Neta [7] and Bi et al. [8,9] are pioneers in developing optimal eighth-order methods. Recently, new families of iterative methods of optimal eighth-order have been published in [10–12]. A good review of optimal and no optimal iterative schemes of different orders of convergence can be found in [13].

Recently, Li et al. [1] have developed a new family of three-point methods based on modification of Chebyshev–Halley's scheme. It is worth mentioning that the methods of this family are not optimal in the sense of Kung–Traub's conjecture, since

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it consumes four functional evaluations per iteration having convergence order five, and a particular member of the family has convergence order six. This family, denoted by LLK5, has the following iterative expression (indices are dropped for simplicity).

$$\begin{cases} y = x - \frac{f(x)}{f'(x)}, \\ z = x - \left(1 + \frac{f(y)}{f(x)-2\beta f(y)}\right)\frac{f(x)}{f'(x)}, \\ \hat{x} = z - \frac{f(z)}{f'(x)+\tilde{f}''(x)(z-x)}, \end{cases} \tag{1.1}$$

where β is a real parameter and $\tilde{f}''(x) = \frac{2f(y)f'(x)^2}{f(x)^2}$. Moreover, its error equation is

$$\hat{e} = 2(\beta-1)c_2^2c_3e^5 + \left((4\beta^2-14\beta+9)c_2^3c_3 + (8\beta-7)c_2c_3^2\right)e^6 + O(e^7), \tag{1.2}$$

being $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2,3,\ldots, e = x-\alpha$ and α a root of $f(x) = 0$.

It can easily be observed that for $\beta = 1$, the first two steps result Ostrowski's method, and, in addition, the convergence order becomes six. In this work, we attempt to derive an optimal three-point method without memory from (1.1) by changing the denominator of the last step. To this end, we suitably approximate $f'(z)$ in the third step instead of $f'(x) + \tilde{f}''(x)(z-x)$ which has been computed by Li et al. [1]. Our approximate is based on Newton–Hermite interpolation at the given data $f(x), f'(x), f(y)$, and $f(z)$. This optimal eighth-order scheme is a particular case of a sixth-order iterative family, depending on parameter β.

Thereafter, we will analyze the stability of the elements of this class of iterative schemes on quadratic polynomials, in terms of the asymptotic behavior of their fixed points and also by using the associated parameter planes, that will allow us to find the most stable elements of the family, under a numerical point of view.

Now, we are going to recall some dynamical concepts of complex dynamics (see [14]) that we use in this work. Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_f \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_f) = z_f$. A *periodic point* z of period $p > 1$ is a point such that $R^p(z) = z$ and $R^k(z) \neq z$, for $k < p$. A *pre-periodic point* is a point z that is not periodic but there exists a $k > 0$ such that $R^k(z)$is periodic. A *critical point* z^* is a point where the derivative of the rational function vanishes, $R'(z^*) = 0$. Moreover, a fixed point z_f is called *attractor* if $|R'(z_f)| < 1$, *superattractor* if $|R'(z_f)| = 0$, *repulsor* if $|R'(z_f)| > 1$ and *parabolic* if $|R'(z_f)| = 1$. So, a superattracting fixed point is also a critical point.

On the other hand, the *basin of attraction* of an attractor $\alpha \in \hat{\mathbb{C}}$ is defined as the set of starting points whose orbits tend to α:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : \ R^n(z_0) \to \alpha, \ n \to \infty\}.$$

The *Fatou set* of the rational function $R, \mathcal{F}(R)$ is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complementary set in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That is, the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The rest of the paper is organized as follows. In Section 2 we present our family of iterative methods, prove the rate of convergence and give the asymptotic error. A particular member of the family is an optimal and very stable eighth-order scheme. In Section 3 we analyze the dynamical behavior of the family, studying the fixed and critical points of the rational function associated to the family on quadratic polynomials. This analysis allows us to obtain some elements of the family with good stability properties. The numerical study presented in Section 4 confirm the theoretical results and allows us to compare our methods with other known ones. The paper finishes with some conclusions and the references used in it.

2. Improved methods and convergence analysis

The main object of this section is to modify method (1.1) so that it has optimal convergence order eight: it must use only four function evaluations. We keep on the first two steps of (1.1), which lead a parametric family of iterative schemes of order three for any value of parameter, and we modify the third step. It is sufficient to find a suitable approximate for $f'(z)$ in the denominator of the third step. Although there are some different and effective approaches for this approximation, we prefer to use the Hermite-Newton interpolation method. Suppose $f(x), f'(x), f(y)$, and $f(z)$ are available. Then, the interpolation polynomial is given by

$$H_3(t) = f(z) + (t-z)f[z,y] + (t-z)(t-y)f[z,y,x] + (t-z)(t-y)(t-x)f[z,y,x,x], \tag{2.1}$$

where

$$f[x_0, x_1, \ldots, x_{k-1}, x_k] = \begin{cases} \frac{f[x_1,\ldots,x_k]-f[x_0,\ldots,x_{k-1}]}{x_k-x_0}, & x_0 \neq x_k, \\ \frac{f^{(k)}(x)}{k!}, & x_0 = \cdots = x_k (= x), \end{cases}$$