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# On developing a higher-order family of double-Newton methods with a bivariate weighting function

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A B S T R A C T

A high-order family of two-point methods costing two derivatives and two functions are developed by introducing a two-variable weighting function in the second step of the classical double-Newton method. Their theoretical and computational properties are fully investigated along with a main theorem describing the order of convergence and the asymptotic error constant as well as proper choices of special cases. A variety of concrete numerical examples and relevant results are extensively treated to verify the underlying theoretical development. In addition, this paper investigates the dynamics of rational iterative maps associated with the proposed method and an existing method based on illustrated description of basins of attraction for various polynomials.



## 1. Introduction

A large number of high-order multipoint methods for a given nonlinear equation $f(x) = 0$ have been developed since Traub [29] initiated the qualitative as well as the quantitative analyses of iterative methods in the 1960s. Petković et al. [25] recently collected and updated the state of the art of multipoint methods. Other works on multipoint methods can be found in [3–5,11,13,14,16,20,24,27]. The principal aim of this paper is to design a family of high-order methods costing only two derivatives and two functions. Described below in (1.1) is the well-known two-point fourth-order double-Newton method [15,29], which is a two-step Newton's method utilizing two derivatives and two functions:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \end{cases} \tag{1.1}$$

This method is only fourth-order. One can get fourth order methods requiring less information.

Among higher-order methods requiring only two derivatives and two functions, we find several three-point sixth-order methods in [5,23,31,20], being respectively shown in (1.2)–(1.5) below.

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - J_f(x_n)\cdot\frac{f(x_n)}{f'(x_n)}, \quad J_f(x_n) = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{a(z_n-x_n)(z_n-y_n)+\frac{3}{2}J_f(x_n)f'(y_n)+(1-\frac{3}{2}J_f(x_n))f'(x_n)}, \quad a \in \mathbb{R}, \end{cases} \tag{1.2}$$

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$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} = z_n - \frac{f'(x_n)+f'(y_n)}{3f'(y_n)-f'(x_n)} \cdot \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{1.3}$$

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{9-5s}{10-6s}\frac{f(x_n)}{f'(y_n)}, \quad s = \frac{f'(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{a+bs}{c+ds+rs^2} \cdot \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{1.4}$$

where $a = (5c+3d+r)/2, b = (r-3c-d)/2, c+d+r \neq 0, a,b,c,d,r \in \mathbb{R}$.

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{1+\beta u}{1+(\beta-2)u}\frac{f(y_n)}{f'(x_n)} = y_n - G_f(u)\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{1-u}{1-3u} \cdot \frac{f(z_n)}{f'(x_n)}. \end{cases} \tag{1.5}$$

where $u = \frac{f(y_n)}{f(x_n)}, G_f(u) = \frac{u(1+\beta u)}{1+(\beta-2)u}, \beta \in \mathbb{R}$.

**Definition 1.1** (*Error equation, asymptotic error constant, order of convergence*). Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence of numbers converging to $\alpha$. Let $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots$. If constants $p \geqslant 1, c \neq 0$ exist in such a way that $e_{n+1} = ce_n^p + O(e_n^{p+1})$ called the *error equation*, then $p$ and $\eta = |c|$ are said to be the *order of convergence* and the *asymptotic error constant*, respectively. It is easy to find $c = \lim_{n\to\infty}\frac{e_{n+1}}{e_n^p}$. Some authors call $c$ the asymptotic error constant.

Three-point methods (1.2)–(1.5) possess rather complicated structures, as compared with two-point methods like (1.1). Among the existing methods requiring two derivatives and two functions, two-point methods of order higher than four are rarely found. As a result, this rareness gives us a strong motivation to design less complicated higher-order two-point methods using two derivatives and two functions. In this paper, our special attention is paid to the development of a general class of two-point higher-order extended double-Newton methods. To this end, by introducing a two-variable weighting function in the second step of (1.1), we propose a higher-order family of two-point methods in the following form:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - K_f(s,u) \cdot \frac{f(y_n)}{f'(y_n)}, \end{cases} \tag{1.6}$$

where the weighting function $K_f : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic[26] in a neighborhood of $(1,0)$ with $s = \frac{f'(y_n)}{f'(x_n)} = 1 + O(e_n)$ and $u = \frac{f(y_n)}{f(x_n)} = O(e_n)$. In view of the fact that $s - 1 = O(e_n), u = O(e_n)$, Taylor series expansion of $K_f(s,u)$ about $(1,0)$ up to terms of several order in each variable will play an essential role in designing two-point sixth-order methods costing two derivatives and two functions.

Note that proposed scheme (1.6) requires four new function evaluations for $f(x_n), f(y_n), f'(x_n), f'(y_n)$. In Section 2, methodology and analysis is described for a new family of sixth-order methods with appropriate forms of $K_f$. Section 3 investigates some special cases of $K_f(s,u)$, Section 4 discusses the extraneous fixed points, while Section 5 presents numerical experiments and concluding remarks.

## 2. Method development and convergence analysis

This section deals with the main theorem and its proof describing the methodology and convergence behavior on iterative scheme (1.6).

**Theorem 2.1.** *Assume that $f : \mathbb{C} \to \mathbb{C}$ has a simple root $\alpha$ and is analytic [1] in a region containing $\alpha$. Let $\Delta = f'(\alpha)$ and $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \ldots$. Let $x_0$ be an initial guess chosen in a sufficiently small neighborhood of $\alpha$. Let $K_f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic in a neighborhood of $(1,0)$. Let $K_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}}{\partial s^i \partial u^j}K_f(s,u)|_{(s=1,u=0)}$ for $0 \leqslant i,j \leqslant 4$. If $K_{00} = 1, K_{01} = K_{10} = 0, K_{20} = \frac{3+K_{02}}{4}, K_{11} = 1 + K_{02}$, $K_{12} = \frac{1}{2}K_{03} + 2(K_{21} - 2K_{30} - 1)$ are satisfied, then iterative scheme (1.6) defines a family of two-point sixth-order methods satisfying the error equation below: for $n = 0, 1, 2, \ldots$,*

$$e_{n+1} = \left\{c_2^2c_4 - \frac{(3+K_{02})}{4}c_2c_3^2 + c_2^3c_3\left(2K_{21} - 8K_{30} - \frac{1}{2}K_{03} - 9\right) + c_2^5\phi\right\}e_n^6 + O(e_n^7), \tag{2.1}$$

*where $\phi = 8K_{31} + 2K_{13} - 4K_{22} - 16K_{40} - K_{04} + 14$.*

**Proof.** Taylor series expansion of $f(x_n)$ about $\alpha$ up to 6th-order terms with $f(\alpha) = 0$ leads us to: