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On parameterized generalized skew-Hermitian triangular splitting iteration method for singular and nonsingular saddle point problems ☆

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ABSTRACT

Recently, Krukier et al. (2014) and Dou et al. (2014) have studied the generalized skew-Hermitian triangular splitting (GSTS) iteration method for nonsingular and singular saddle point problems, respectively. In this paper, we further extend the GSTS method to a parameterized GSTS (PGSTS) method for solving non-Hermitian nonsingular and singular saddle point problems. By singular value decomposition technique, we derive conditions of the new iterative method for guaranteeing the convergence for non-Hermitian nonsingular saddle point problems and its semi-convergence for singular saddle point problems, respectively. In addition, the choice of the acceleration parameters in a practical manner is studied. Numerical experiments are provided, which further confirm our theoretical results and show the new method is feasible and effective for non-Hermitian nonsingular or singular saddle point problems.

1. Introduction

We consider the saddle point problems of the form

$$\mathcal{A}u \equiv \begin{pmatrix} M & E \\ -E^* & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1.1}$$

where $M \in \mathbb{C}^{m\times m}$ is Hermitian positive definite, $E \in \mathbb{C}^{m\times n}$ is a rectangular matrix, $x, f \in \mathbb{C}^m$ and $y,\ g \in \mathbb{C}^n$ are given vectors, with $n \leqslant m$. We denote the conjugate transposes of the matrix E and the vector x by E^* and x^*, respectively. The saddle point problem (1.1) arises in a wide variety of scientific and engineering applications such as computational fluid dynamics [23], mixed finite element approximation of elliptic partial differential equations [16], optimal control [11], weighted least-squares problems [24], electronic networks computer graphics [32] and others, see [13] for a detail.

When the linear system (1.1) is nonsingular, e.g., E is of full column rank, a large variety of methods for solving linear systems of the form (1.1) has been proposed in the literature, including Uzawa-type methods [4,5,15,21], Hermitian and skew-Hermitian splitting (HSS) iteration methods as well as its accelerated variants [2,3,6–8,31], SOR-like method [36,40], RPCG method [9,10], Krylov subspace methods [18,25,26] and so on, see [12] for a survey.

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When E is rank deficient, the linear system (1.1) is singular. There are also many efficient methods for solving this case, including Uzawa-like methods [29,30,35,37,39] and HSS-like methods [1,19,33], Krylov subspace methods [38] and so on.

Recently, Krukier et al. [27] developed a generalized skew-Hermitian triangular splitting (GSTS) iteration method for the nonsingular saddle point problem (1.1). It can be described as follows.

The GSTS method for nonsingular saddle point problems [27]:

$$\begin{cases} y_{k+1} = y_k + \tau B^{-1}[\omega_1 E^* M^{-1}(f - E y_k) + (1-\omega_1)E^* x_k + g], \\ x_{k+1} = (1-\tau)x_k + M^{-1}[E((\omega_2 - \tau)y_k - \omega_2 y_{k+1}) + \tau f], \end{cases} \tag{1.2}$$

where B is a prescribed Hermitian and positive definite matrix.

Dou et al. in [20] further studied the GSTS method for solving the singular saddle point problem (1.1).

The GSTS method for singular saddle point problems[20]:

$$\begin{cases} y_{k+1} = y_k + \tau B^{\dagger}[\omega_1 E^* M^{-1}(f - E y_k) + (1-\omega_1)E^* x_k + g], \\ x_{k+1} = (1-\tau)x_k + M^{-1}[E((\omega_2 - \tau)y_k - \omega_2 y_{k+1}) + \tau f], \end{cases} \tag{1.3}$$

where $B^{\dagger}$ is the Moore–Penrose inverse of the singular matrix B, which satisfies

$$B = BB^{\dagger}B, \quad B^{\dagger} = B^{\dagger}BB^{\dagger}, \quad (BB^{\dagger}) = (BB^{\dagger})^*, \quad (B^{\dagger}B) = (B^{\dagger}B)^*. \tag{1.4}$$

In this paper, based on the GSTS methods proposed in [27,20], we extend it to a novel parameterized generalized skew-Hermitian triangular splitting (PGSTS) iteration method to solve the non-Hermitian nonsingular and the singular saddle point problem (1.1). The choices of the parameters is studied in detail. A feasible method to select the acceleration parameters is provided. The convergence and the semi-convergence of the PGSTS method for non-Hermitian nonsingular and singular saddle point problem, respectively, are discussed. Numerical experiments are given to show the advantages of the PGSTS method for solving both nonsingular and singular saddle point problems.

This paper is organized as follows: In Section 2, we establish the parameterized generalized skew-Hermitian triangular splitting (PGSTS) iteration method for solving non-Hermitian non-singular saddle point problem. In this section, we not only deduce the conditions for guaranteeing its convergence, but also discuss the choice of the acceleration parameters in a practical manner. In Section 3, the PGSTS method is further studied for solving singular saddle point problem by using singular or non-singular preconditioner. In Section 4, we provide some numerical experiments for the comparison of the PGSTS with the GSTS and the GSOR methods, which shows the PGSTS method is feasible and effective. Finally, some conclusions are made in Section 5.

2. PGSTS method for nonsingular saddle point problems

In this section, we assume that the matrix E defined in (1.1) is of full column rank. In this case, it is known that the linear system (1.1) is non-singular.

2.1. The PGSTS method

Splitting the saddle point matrix $\mathcal{A}$ in (1.1) into its Hermitian and skew-Hermitian parts as

$$\mathcal{A}_H = \frac{1}{2}(\mathcal{A} + \mathcal{A}^*) = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_S = \frac{1}{2}(\mathcal{A} - \mathcal{A}^*) = \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}. \tag{2.1}$$

Let $\mathcal{K}_L$ and $\mathcal{K}_U$ be the strictly lower-triangular and the strictly upper-triangular parts of the matrix $\mathcal{A}_S$, respectively. Note that $\mathcal{K}_L = -\mathcal{K}_U^*$ with

$$\mathcal{K}_L = \begin{pmatrix} 0 & 0 \\ -E^* & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{K}_U^* = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}. \tag{2.2}$$

Define

$$\Omega := \begin{pmatrix} \tau I_m & 0 \\ 0 & \gamma I_n \end{pmatrix} \quad \text{and} \quad \mathcal{B}_c := \begin{pmatrix} M & 0 \\ 0 & B \end{pmatrix} \tag{2.3}$$

with τ and γ being two positive parameters. Here I_m and I_n are the identity matrices of order m and n, respectively, $B \in \mathbb{C}^{n\times n}$ is an Hermitian positive definite matrix. Based on the GSTS iteration method [27], we present the following parameterized generalized skew-Hermitian triangular splitting (PGSTS) iteration method for saddle point problem (1.1):

$$u_{k+1} = \mathcal{G}(\omega_1, \omega_2, \tau, \gamma)u_k + \Omega\mathcal{B}(\omega_1, \omega_2)^{-1}b. \tag{2.4}$$