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A few remarks on orthogonal polynomials ☆

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ABSTRACT

Knowing a sequence of moments of a given, infinitely supported, distribution we obtain quickly: coefficients of the power series expansion of monic polynomials $\{p_n\}_{n\geqslant 0}$ that are orthogonal with respect to this distribution, coefficients of expansion of x^n in the series of $p_j, j \leqslant n$, two sequences of coefficients of the 3-term recurrence of the family of $\{p_n\}_{n\geqslant 0}$, the so called "linearization coefficients" i.e. coefficients of expansion of $p_n p_m$ in the series of $p_j, j \leqslant m+n$.

Conversely, assuming knowledge of the two sequences of coefficients of the 3-term recurrence of a given family of orthogonal polynomials $\{p_n\}_{n\geqslant 0}$, we express with their help: coefficients of the power series expansion of p_n, coefficients of expansion of x^n in the series of $p_j, j \leqslant n$, moments of the distribution that makes polynomials $\{p_n\}_{n\geqslant 0}$ orthogonal.

Further having two different families of orthogonal polynomials $\{p_n\}_{n\geqslant 0}$ and $\{q_n\}_{n\geqslant 0}$ and knowing for each of them sequences of the 3-term recurrences, we give sequence of the so called "connection coefficients" between these two families of polynomials. That is coefficients of the expansions of p_n in the series of $q_j, j \leqslant n$.

We are able to do all this due to special approach in which we treat vector of orthogonal polynomials $\{p_j(x)\}_{j=0}^n$ as a linear transformation of the vector $\{x^j\}_{j=0}^n$ by some lower triangular $(n+1)\times(n+1)$ matrix $\mathbf{\Pi}_n$.

1. Introduction and notation

Let us first make some remarks concerning notation. $\alpha, \beta, \ldots$ will denote positive measures on the real line. We will assume that all of these measures have infinite supports. In order to be able to sometimes use probabilistic notation we will assume that all considered measures are normalized. Integrals of an integrable function f with respect to the measure α will be denoted by either of the following denotations:

$$\int f(x)d\alpha(x), \int f d\alpha, \mathbb{E}f, \mathbb{E}f(Z), \mathbb{E}_\alpha f(Z),$$

depending on the context and the need to specify details. In the above formulae, Z denotes random variable with the distribution α. Probability theory ensures that Z always exist.

Matrices and vectors (always columns) will be generally denoted by the bold type letters. The most important vector and matrix are $\mathbf{X}_n = (1, x, \ldots, x^n)^T$ (T-transposition) and

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$$\mathbf{M}_n(\alpha) = \left[m_{i+j}(\alpha)\right]_{j,i=0,\dots,n}, \tag{1.1}$$

where $m_n(\alpha) = \int x^n d\alpha(x)$. In other words $\mathbf{M}_n(\alpha) = \mathbb{E}_\alpha \mathbf{X}_n \mathbf{X}_n^T$. Matrices of this form, that is having the same elements on counter diagonals, are called Hankel matrices.

Definition 1. We will say that the moment problem is determinate if there exists only one measure α that generates the moment sequence $\{m_n(\alpha)\}_{n\geqslant 0}$. Otherwise we say that the moment problem is indeterminate.

Remark 1. There exist sufficient criteria allowing to check if the moment problem is determinate or not. For example Carleman's criterion states that if $\sum_{n\geqslant 0} m_{2n}^{1/2n} < \infty$, then the moment problem is indeterminate. Else, if $\int \exp(|x|)d\alpha(x) < \infty$, then the moment problem is determinate (for details see e.g. [14] or [15]).

In the sequel we will assume generally that our moment problem is determinate.

$(\mathbf{A})_{j,k}$ will denote (j,k)-th entry of the matrix $\mathbf{A}$.

Infinite support assumption ensures that for every n one can always find $n+1$ linearly independent vectors of the form $(1, x_k, \dots, x_k^n)^T$, where $x_k \in supp\alpha$. Besides we know that if $supp\alpha$ is infinite then matrices $\mathbf{M}_n(\alpha)$ are non-singular for every n. Let us remark immediately that the matrix $\mathbf{M}_n$ is the main submatrix of the matrix $\mathbf{M}_{n+1}$. We also define sequence

$$\Delta_n(\alpha) = \det \mathbf{M}_n(\alpha), \tag{1.2}$$

$n \geqslant 1$, of determinants of matrices $\mathbf{M}_n(\alpha)$ and let us also introduce vectors consisting of successive moments

$$\mathbf{m}_n^T(\alpha) = (1, \dots, m_n(\alpha)).$$

Vector $\mathbf{m}_n(\alpha)$ is the first column of the matrix $\mathbf{M}_n(\alpha)$.

In order to avoid repetition of assumption we will assume that matrices $\mathbf{M}_n(\alpha)$ exist for all $n \geqslant 0$. In other words we assume that all moments of the measure α exist. Obviously $(0,0)$ entry of the matrix $\mathbf{M}_n$ is equal to 1.

We know that given measure α, such that all moments exist, one can define the set of polynomials $\{p_n(x,\alpha)\}_{n\geqslant -1}$ with $p_{-1}(x,\alpha) = 0, p_0(x,\alpha) = 1$, such that p_n is of degree n and satisfying for $n+m \neq -2$ the following relationship:

$$\int p_n(x,\alpha)p_m(x,\alpha)d\alpha(x) = \delta_{n,m},$$

where $\delta_{n,m}$ denotes Kronecker's delta. Moreover if we declare that all leading coefficients of the polynomials $p_n(x,\alpha)$ are positive then the coefficients $\pi_{n,i}(\alpha)$ of the expansion

$$p_n(x,\alpha) = \sum_{i=0}^{n} \pi_{n,i}(\alpha)x^i \tag{1.3}$$

are defined uniquely by the measure α. According to our convention, later we will drop dependence on α if the measure α is clearly specified.

Let us define vectors $\mathbf{P}_n(x) = (p_0(x), \dots, p_n(x))^T$ and the lower triangular matrix $\mathbf{\Pi}_n$ with entries $\pi_{i,j}$ if $i \geqslant j$ and 0 otherwise. We obviously have:

$$\mathbf{P}_n(x) = \mathbf{\Pi}_n \mathbf{X}_n. \tag{1.4}$$

To continue introduction of notation, let $\lambda_{n,i}(\alpha)$ denote coefficients in the following expansions:

$$x^n = \sum_{i=0}^{n} \lambda_{n,i}(\alpha)p_i(x,\alpha). \tag{1.5}$$

Consequently let us introduce lower triangular matrices $\mathbf{\Lambda}_n$ with entries $\lambda_{i,j}$ if $i \geqslant j$ and 0 otherwise.

We obviously have:

$$\mathbf{X}_n = \mathbf{\Lambda}_n \mathbf{P}_n(x), \mathbf{\Pi}_n \mathbf{\Lambda}_n = \mathbf{\Lambda}_n \mathbf{\Pi}_n = \mathbf{I}_n, \tag{1.6}$$

where $\mathbf{I}_n$ denotes $(n+1) \times (n+1)$ identity matrix.

As polynomials $\{p_n\}$ are orthonormal, there exist two number sequences $\{a_n\}, \{b_n\}$ such that polynomials $\{p_n\}$ satisfy the following 3-term recurrence:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \tag{1.7}$$

with $a_0 = 0$ and $n \geqslant 0$. We know also that

$$a_n = \frac{\pi_{n-1,n-1}}{\pi_{n,n}}, b_n = \int x p_n^2(x) d\alpha(x), \tag{1.8}$$

consequently that $b_0 = m_1$. For details see e.g. [1] or [12].