Applied Mathematics and Computation 252 (2015) 229–239

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A hybrid finite difference scheme for pricing Asian options

Zhongdi Cen *, Aimin Xu, Anbo Le

Institute of Mathematics, Zhejiang Wanli University, Ningbo 315100, PR China

A R T I C L E I N F O

Keywords:
Asian option
Partial differential equation
Central difference method
Midpoint upwind scheme
Crank–Nicolson method

A B S T R A C T

In this paper we apply a hybrid finite difference scheme to evaluate the prices of Asian call options with fixed strike price. We use the Crank–Nicolson method to discretize the time variable and a hybrid finite difference scheme to discretize the spatial variable. The hybrid difference scheme uses the central difference approximation whenever the mesh points are sufficiently away from the left-hand side of the domain to ensure the stability of the scheme; otherwise a midpoint upwind difference scheme is used. The matrix associated with the discrete operator is an M-matrix, which ensures that the spatial discretization scheme is maximum-norm stable. It is proved that the scheme is second-order convergent with respect to both time and spatial variables. Numerical experiments support these theoretical results.

1. Introduction

Asian options are derivative securities whose payoffs depend on the averaging prices of the underlying asset over some time period. Such options are used by traders who are interested to hedge against the average price of a commodity over a period rather than the price at the expiration date. The price of the Asian option is less subject to price manipulation. Hence such options are particularly useful for business involved in trading on thinly traded commodities. However, pricing and hedging Asian options is difficult, especially for options depending on arithmetic averaging. Generally, no closed-form solutions are available. Thus, a variety of numerical pricing techniques have been proposed [3].

In this paper we focus on pricing arithmetics average Asian call options with fixed strike price. Suppose that the underlying asset price S_t follows geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where r is the risk-free interest rate, σ is the volatility, and B_t is a standard Brownian motion under the risk-neutral measure $\mathbb{P}$. Let $I(t)$ represent the underlying asset price running sum as

$$I(t) = \int_0^t S(\tau)\mathrm{d}\tau,$$

then the average is given by $A(t) = I(t)/t$. Thus the price $C(S, I, t)$ of the continuous arithmetics average Asian call option with fixed strike price satisfies the following two-dimensional PDE [11,21]

$$-\frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} - S\frac{\partial C}{\partial I} + rC = 0 \tag{1.1}$$

* Corresponding author.
E-mail address: finite@zwu.edu.cn (Z. Cen).

http://dx.doi.org/10.1016/j.amc.2014.12.007

with the final condition

$$C(S,I,T) = \max\left(\frac{I}{T} - E, 0\right), \tag{1.2}$$

where T is the expiry date and E is the strike price. This PDE is a so-called degenerate parabolic problem with several degeneracies. As is well-known in computational fluid dynamics, standard central difference method for the convective term will generate spurious oscillations [1,22].

There are several methods in the open literature for solving the two-dimensional PDE (1.1) and (1.2) directly. Zvan et al. [22] use flux limiting techniques to price Asian options. Hugger [10] investigates the dependence of the numerical methods on the various degeneracies and the effectiveness of the artificial viscosity numerical method for pricing a fixed strike Asian option. Bermúdez et al. [2] present an iterative algorithm combined with higher order Lagrange–Galerkin methods for pricing Amerasian options. Cen et al. [4] develop a numerical method which is deduced by combining an alternating direction technique and central difference scheme on a piecewise uniform mesh. Tangman et al. [19] apply the exponential time integration scheme with a dimensional splitting strategy to price Asian options under a variety of pricing models.

Note that the problem (1.1) and (1.2) is a two-dimensional PDE which leads to greater computational costs. This motivates the reduction of the problem into one-dimensional problem. By making the change of variables

$$x = \frac{E - I/T}{S}, \quad C(S,I,t) = Sv(x,t), \tag{1.3}$$

Rogers and Shi [17] have reduced the two-dimensional PDE (1.1) and (1.2) to the following one-dimensional PDE

$$-\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \left(\frac{1}{T} + rx\right)\frac{\partial v}{\partial x} = 0, \tag{1.4}$$

$$v(x,T) = \max(-x, 0). \tag{1.5}$$

Note that in Geman and Yor [9], a formula is obtained for the case $I \geqslant ET$ as

$$C(S,I,t) = \frac{S}{rT}\left(1 - e^{-r(T-t)}\right) + \left(\frac{I}{T} - E\right)e^{-r(T-t)}.$$

Then, by making the change of variables as in (1.3), we have

$$v(x,t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) - xe^{-r(T-t)} \quad \text{for } x \leqslant 0. \tag{1.6}$$

Hence we only consider the solution of the PDE (1.4) and (1.5) for $x > 0$. From (1.6) we obtain the left hand boundary condition

$$v(0,t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right).$$

The right hand boundary condition for $x \to +\infty$ can be obtained by observing that we only consider the case $I < ET$, so $S \to 0$ for $x \to +\infty$. For $S \to 0$ the option is not exercised rendering it's value to be 0, i.e.,

$$\lim_{x\to+\infty} v(x,t) = 0.$$

Thus, we will focus on the following PDE

$$\begin{aligned}
&-\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \left(\frac{1}{T} + rx\right)\frac{\partial v}{\partial x} = 0,\\
&v(x,T) = 0,\\
&v(0,t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right),\\
&\lim_{x\to+\infty} v(x,t) = 0.
\end{aligned}$$

For applying the numerical method we need truncate the infinite domain $(0, +\infty)$ into $(0, X)$ with an appropriate choice of X. The boundary condition at $x = X$ is chosen to be $v(X,t) = 0$. Therefore, in the remaining of this paper we will consider the following PDE [17]

$$-\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \left(\frac{1}{T} + rx\right)\frac{\partial v}{\partial x} = 0, \quad (x,t) \in (0,X) \times [0,T), \tag{1.7}$$

$$v(x,T) = 0, \quad x \in [0,X], \tag{1.8}$$

$$v(0,t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right), \quad t \in [0,T), \tag{1.9}$$