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# Steady states solutions of Allen–Cahn equation by computer algebra

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ABSTRACT

In this paper, we consider the analytical solution and numerical approximation of the celebrated steady states Allen–Cahn equation. We present a computer code to solve and plot the solutions using *Maxima* software.



## 1. Introduction

Consider the boundary value problem of the form

$$\begin{cases} -\varphi'' = \varphi - \varphi^3 & \text{in } (a,b) \\ \varphi(a) = \alpha, & \varphi(b) = \beta. \end{cases} \tag{1.1}$$

where $\alpha, \beta$ are given constants, and the function $\varphi$ is often called the phase field function or an order parameter.

Eq. (1.1) is the one-dimensional stationary case of the well-known Allen–Cahn equation

$$\begin{cases} \varphi_t(x,t) = \xi^2 \varphi_{xx}(x,t) + \varphi - \varphi^3 & \text{in } (a,b) \times (0,T) \\ \varphi(a,t) = \alpha, \quad \varphi(b,t) = \beta, & \text{for } t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), & x \in (a,b). \end{cases}$$

The Allen–Cahn equation, which has its origin in the modeling of phase transition or the equation which arises in the study of stationary waves for the nonlinear Schrdinger equation have been a subject of extensive research for many years. It provides a well-established framework for the mathematical description to free boundary problems for phase transitions. The physical applications of the Allen–Cahn system are numerous, see for example [4,5,8], due to which a lot of interest has been shown in numerical as well as analytical solutions of Allen–Cahn equation using computer algebra (see Fig. 1).

An important feature of the Allen–Cahn equation is that can be viewed as the gradient flow of the energy functional

$$F(\varphi) = \frac{1}{2}\int_a^b \varphi'^2 dx + \frac{1}{4}\int_a^b (\varphi^2 - 1)^2 dx. \tag{1.2}$$

In this paper, an alternative approach by using the computer algebra system *Maxima* [7] is proposed to provide an efficient analytical and numerical solution of the Allen–Cahn Eq. (1.1).

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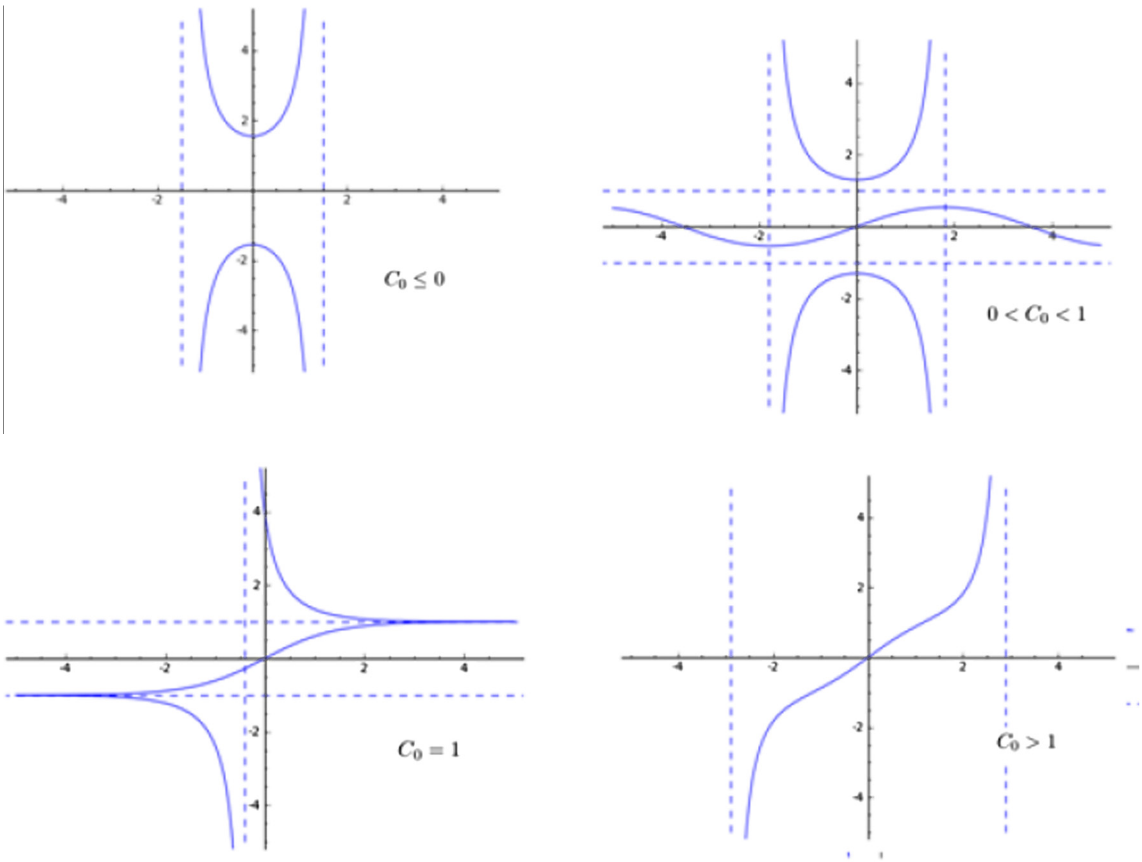


**Fig. 1.** Profile of solutions for different values of $C_0$.

The paper is organized as follows. In Section 2 we study the general solution of problem (1.1). In Section 3 is devoted to background material of the Ritz method that will be used to problem (1.1). In Section 4, some numerical examples are given to illustrate the accuracy of the method. Section 5 consists of our brief conclusions.

## 2. General solution

In this section, we will study the general solution of equation

$$-\varphi'' = \varphi - \varphi^3. \tag{2.1}$$

For this, we use the *Maxima* and *Sage*, an international mathematical open softwares.

*Maxima* is an exchanged computer algebra system with great ability of symbolic operation, numerical calculation, coping with graphics, etc. Its powerful functions library and unique interior programming language provide scientific calculation.

*(1) Constant solutions*

$\varphi \equiv -1, \varphi \equiv 0$ and $\varphi \equiv 1$ are the only constant solutions of (2.1).

*(2) $\varphi' \neq 0$ in a interval*

Multiplying (2.1) by $\varphi'$ and integrating, we have

$$\varphi' = \pm\sqrt{\frac{1}{2}(\varphi^4 - 2\varphi^2 + C_0)},$$

with $C_0 \in \mathbb{R}$. Then

$$x = \pm\sqrt{2}\int \frac{1}{\sqrt{\varphi^4 - 2\varphi^2 + C_0}}\, d\varphi. \tag{2.2}$$