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On a functional equation of trigonometric type

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ABSTRACT

In this paper, we study the functional equation, $f(x+y)-f(x)f(y)=d\sin x\sin y$. Some generalizations of the above functional equation are also considered.

1. Introduction

In the fall of 1940, Ulam gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms (cf. [15]):

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a function $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$?

If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

Hyers was the first mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces (see [5]). This result of Hyers is stated as follows:

Theorem 1.1. *Let $f:E_1\to E_2$ be a function between Banach spaces such that*

$$\|f(x+y)-f(x)-f(y)\|\leqslant\delta \tag{1.1}$$

for some $\delta>0$ and for all $x,y\in E_1$. Then the limit

$$A(x)=\lim_{n\to\infty}2^{-n}f(2^n x) \tag{1.2}$$

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exists for each $x \in E_1$, *and* $A : E_1 \to E_2$ *is the unique additive function such that*

$$\|f(x) - A(x)\| \leqslant \delta$$

for every $x \in E_1$. *Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in E_1$, *then the function* A *is linear.*

Taking this result into consideration, the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers–Ulam stability on $(E_1; E_2)$ if for every $\delta > 0$ there exist a bounded subset M of E_2 such that for every function $f : E_1 \to E_2$ satisfying inequality (1.1) there exists an additive function $A : E_1 \to E_2$ such that $f(x) - A(x) \in M$ for every $x \in E_1$, i.e., the difference $f - A$ is uniformly bounded.

For a broad study of the Hyers–Ulam stability for a large variety of functional equations the reader is referred to [3,4,6,8,10,12].

In this paper, we will present some results concerning the solution as well as the Hyers–Ulam stability of the functional equation

$$f(x+y) - f(x)f(y) = d \sin x \sin y, \tag{1.3}$$

where d is a real constant less than -1. Moreover, we introduce some functional equations of the form $f(x+y) + \lambda f(x)f(y) = \Phi(x,y)$ and then we investigate their stability properties (see [14]).

2. Preliminaries

In 2003, Butler [2] posed the following question:

Problem 2.1. Show that for $d < -1$ there are exactly two solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation (1.3).

In 2004, Rassias answered this question by proving the following theorem (see [11]):

Theorem 2.1. *Let* $d < -1$ *be a constant. The functional equation (1.3) has exactly two solutions in the class of functions* $f : \mathbb{R} \to \mathbb{R}$. *More precisely, if a function* $f : \mathbb{R} \to \mathbb{R}$ *is a solution of Eq. (1.3), then* f *has one of the forms*

$$f(x) = c \sin x + \cos x \quad \text{and} \quad f(x) = -c \sin x + \cos x,$$

where $c = \sqrt{-d-1}$.

Corollary 2.2. *Let* $d < -1$ *be a constant. The functional equation*

$$g\left(x + y - \frac{\pi}{2}\right) - g(x)g(y) = d \cos x \cos y \tag{2.1}$$

has exactly two solutions in the class of functions $g : \mathbb{R} \to \mathbb{R}$. *More precisely, if a function* $g : \mathbb{R} \to \mathbb{R}$ *is a solution of Eq. (2.1), then* g *has one of the forms*

$$g(x) = \sin x + c \cos x \quad \text{and} \quad f(x) = \sin x - c \cos x,$$

where $c = \sqrt{-d-1}$.

Proof. Replacing in (2.1) $x,\ y$ by $\frac{\pi}{2} - x$ and $\frac{\pi}{2} - y$ respectively, we get

$$g\left(\frac{\pi}{2} - x - y\right) - g\left(\frac{\pi}{2} - x\right)g\left(\frac{\pi}{2} - y\right) = d \sin x \sin y.$$

Now the function $f(x) = g(\frac{\pi}{2} - x)$ satisfies the functional Eq. (1.3). By Theorem 2.1,

$$g\left(\frac{\pi}{2} - x\right) = \pm c \cos x + \sin x$$

and the conclusion follows by replacing back x by $\frac{\pi}{2} - x$. □

Proof of Theorem 2.1 (*M.Th. Rassias*). Replacing x with $x + z$ in (1.3), we get

$$f(x+y+z) - f(x+z)f(y) - d \sin(x+z) \sin y = 0 \tag{2.2}$$

for all $x, y, z \in \mathbb{R}$. Similarly, if we replace y with $y + z$ in (1.3), then we get

$$f(x+y+z) - f(x)f(y+z) - d \sin x \sin(y+z) = 0 \tag{2.3}$$

for all $x, y, z \in \mathbb{R}$.

It follows from (2.2) and (2.3) that

$$f(x)f(y+z) - f(x+z)f(y) + d \sin x \sin(y+z) - d \sin(x+z) \sin y = 0$$