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On the convergence of a Damped Secant method with modified right-hand side vector ☆

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ABSTRACT

We present a convergence analysis for a Damped Secant method with modified right-hand side vector in order to approximate a locally unique solution of a nonlinear equation in a Banach spaces setting. In the special case when the method is defined on $\mathbb{R}^i$, our method provides computable error estimates based on the initial data. Such estimates were not given in relevant studies such as (Herceg et al., 1996; Krejić, 2002). Numerical examples further validating the theoretical results are also presented in this study.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a open convex subset D of a Banach space $\mathbb{X}$ with values in a Banach space $\mathbb{Y}$.

Many problems from Computational Sciences and other disciplines can be brought in a form similar to Eq. (1.1) using Mathematical Modeling [3,7,8,11,14]. For example in data fitting, we have $\mathbb{X} = \mathbb{Y} = \mathbb{R}^i, i$ is number of parameters and observations.

The solution of (1.1) can rarely be found in closed form. That is why most solution methods for these equations are usually iterative. In particular, the practice of Numerical Analysis for finding such solutions is essentially connected to Newton-like methods [3,5–7,14–16]. The study about convergence matter of iterative procedures is usually centered on two types: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures; while the local one is, based on the information around a solution, to find estimates of the radii of the convergence balls. Local and semilocal convergence of Newton-like methods as well as an error analysis for such methods can be found in [1,2,5,7,9,10].

In the present paper, we study the convergence of the Damped Secant method defined by

$$x_{n+1} = x_n - A^{-1}(I - \alpha_n([x_{n-1}, x_n; F] - A))F(x_n), \quad \text{for each} \quad n = 0, 1, 2, \dots, \tag{1.2}$$

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where x_{-1} and x_0 are initial points, $A \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}, A^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X}), [x, y; F] \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is a divided difference of order one at the point (x, y) with $x, y \in D$ and α_n is a sequence of real numbers chosen to force convergence of sequence x_n.

If the divided difference $[x, y; F]$ is replaced by $F'(x)$ we get

$$z_{n+1} = z_n - A^{-1}\big(I - \alpha_n\big(F'(z_n) - A\big)\big)F(z_n) \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

The local convergence of the method (1.3) was studied by Krejić and Lužanin [13] (see also [12]) in the case when $\mathbb{X} = \mathbb{Y} = \mathbb{R}^i$.

If $A = [x_{n-1}, x_n; F]$ and $\alpha_n = 0$ for each $n = 0, 1, 2, \ldots$, we obtain the secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1}F(x_n), \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

If $A = [x_{-1}, x_0; F]$ and $\alpha_n = 0$ for each $n = 0, 1, 2, \ldots$, we obtain the modified-secant method

$$x_{n+1} = x_n - [x_{-1}, x_0; F]^{-1}F(x_n), \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

It is well-known that Newton's method

$$x_{n+1} = x_n - F'(x_n)F(x_n), \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

converges quadratically provided that the iteration starts close enough to the solution. However, the cost of a Newton iterate may be very expensive, since all the elements of the Jacobian matrix involved must be computed, as well as the need for an exact slowdown of a system of linear equations using a new matrix for every iterate. As noted in [13] Newton-like method (1.3) uses a modification of the right hand side vector, which is cheaper than the Newton and faster than the modified Newton method. One step of the method requires the solution of a linear system, but the system matrix is the same in all iterations.

Similar problems we have if we use the secant method (1.4) which converges with order $1.618\ldots < 2$. That is why Damped Secant method is a good alternative to all the preceding methods, especially to the method (1.3) considered in [13]. Let us remark that, under similar convergence conditions to secant method, an inverse-free iterative scheme can be found depending on the selection of matrix A. By using a diagonal matrix A, local and semilocal conditions can be satisfied and computational cost is highly reduced, as no matrix inversion or solution of linear system is needed.

We present a local and semilocal convergence analysis for Damped Secant method (1.2). In the local case the radius of convergence can be computed as well as the error bounds on the distances $\|x_n - x^*\|$ for each $n = 0, 1, 2, \ldots$. In the semilocal case, we provide estimates on the smallness of $\|F(x_0)\|$ as well as computable estimates for $\|x_n - x^*\|$ (not given in [12,13] in terms of the Lipschitz constants and other initial data).

We denote by $U(v, \mu)$ the open ball centered at $v \in \mathbb{X}$ and of radius $\mu > 0$. Moreover, by $\overline{U(v, \mu)}$ we denote the closure of $U(v, \mu)$.

The paper is organized as follows. Sections 2 and 3 contain the semilocal and local convergence analysis of Secant-like method (1.2), respectively. The numerical examples are presented in the concluding Section 4.

2. Semilocal convergence

In this section we present the semilocal convergence of Damped Secant method (1.2). We shall use the following conditions:

(C_0) $F : D \subseteq \mathbb{X} \to \mathbb{Y}$ is Fréchet-differentiable and there exists $A \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ such that $A^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ with $\|A^{-1}\| \leqslant a$;

(C_1) There exists $L > 0$ such that for each $x, y \in D$ the Lipschitz condition

$$\|F'(x) - F'(y)\| \leqslant L\|x - y\|,$$

holds;

(C_2) Given $x_0 \in D$, there exist $L_0 > 0, a_0 \geqslant 0$ and $a_1 \geqslant 0$ such that for each $x \in D$ the center-Lipschitz condition

$$\|F'(x) - F'(x_0)\| \leqslant L_0\|x - x_0\|,$$

$\|F'(x_0) - A\| \leqslant a_0$ and $\|A^{-1}(F'(x_0) - A)\| \leqslant a_1$ hold;

(C_3) There exist a divided difference $[x, y; F]$ and $L_1 \geqslant 0$ such that for each $x, y \in D$

$$\|[x, y; F] - F'(x_0)\| \leqslant L_1(\|x - x_0\| + \|y - x_0\|);$$

(C_4) There exists $\alpha \geqslant 0$ and $c > 0$ such that

$$|\alpha_n| \leqslant \alpha \qquad \text{and} \qquad \|x_{-1} - x_0\| \leqslant c, \; x_{-1} \in D.$$

(C_5) There exists $q \in (0, 1)$ such that