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# On the convergence of an optimal fourth-order family of methods and its dynamics

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ABSTRACT

In this paper, we present the study of the semilocal and local convergence of an optimal fourth-order family of methods. Moreover, the dynamical behavior of this family of iterative methods applied to quadratic polynomials is studied. Some anomalies are found in this family be means of studying the dynamical behavior. Parameter spaces are shown and the study of the stability of all the fixed points is presented.



## 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^*$ of equation

$$F(x) = 0, \tag{1.1}$$

where $F$ is a nonlinear operator defined on a convex subset $D$ of a normed space $X$ with values in a normed space $Y$.

A vast number of problems from Applied Sciences including engineering can be solved by means of finding the solutions equations in a form like (1.1) using mathematical modeling [3,17]. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. Except in special cases, the solutions of these equations can be found in closed form. This is the main reason why the most commonly used solution methods are iterative. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypothesis. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$. These are with the study of the dynamical behavior our objectives in this paper.

The dynamical properties related to an iterative method applied to polynomials give important information about its stability and reliability. In recently studies, authors such us Cordero et al. [11,12], Amat et al. [1–4], Gutiérrez et al. [14], Chun et al. [10] and many others [8,17,18] have found interesting dynamical planes, including periodical behavior and others anomalies. One of our main interests in this paper is the study of the parameter spaces associated to a family of iterative methods, which allow us to distinguish between the good and bad methods in terms of its numerical properties.

In this work, we consider the following optimal fourth-order family of methods presented by Behl in [6].

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$$v_n = x_n - \frac{2}{3}\frac{F(x_n)}{F'(x_n)},$$
$$x_{n+1} = x_n - \frac{[(\beta^2 - 22\beta - 27)F'(x_n) + 3(\beta^2 + 10\beta + 5)F'(v_n)]F(x_n)}{2(\beta F'(x_n) + 3F'(v_n))(3(\beta + 1)F'(v_n) - (\beta + 5)F'(x_n))}, \quad (1.2)$$

where $\beta$ is a complex parameter. Notice that every member of this family is an optimal fourth-order iterative method. In this paper, the dynamics of this family applied to an arbitrary quadratic polynomial $p(z) = (z - a)(z - b)$ will be analyzed, characterizing the stability of all the fixed points. The graphic tool used to obtain the parameter space and the different dynamical planes have been introduced by Magreñán in [15,16], but there exist other techniques such us the one given by Chicharro et al. in [9].

The rest of the paper is organized as follows: in Section 2 the study of the semilocal convergence is presented, in Section 3 the local convergence is studied and in Section 4 some of the basic dynamical concepts related to the complex plane are presented, the stability of the fixed points of the family and the dynamical behavior of the family is analyzed, where the parameter space and some selected dynamical planes are presented. Finally, the conclusions drawn to this study are presented in the concluding Section 5.

## 2. Semilocal convergence

The semilocal convergence analysis of method (1.2) is presented in this section by treating it as a Newton-like method [5,19]. There is a plethora of semilocal convergence results for Newton-like methods under various Lipschitz-type conditions. In this section, we present only one such result. We refer the reader to [5] and the references therein for other results, where the idea of this section can also be used to generate semilocal convergence results for the method (1.2).

In this section, we assume that $X$ and $Y$ are Banach spaces. Denote by $U(w,\rho)$, $\overline{U}(w,\rho)$ the open and closed balls in $X$, respectively, with center $w \in X$ and of radius $\rho > 0$.

Next, we present a general semilocal convergence results for approximating a locally unique solution $x^*$ of a nonlinear Eq. (1.1).

**Theorem 1** ([5,19]). *Let $F : D \subset X \to Y$ be Fréchet-differentiable and let $A(x) \in L(X,Y)$ be an approximation of $F'(x)$. Suppose that there exist an open convex subset $D_0$ of $D$, $x_0 \in D_0$, $A(x_0)^{-1} \in L(Y,X)$ and constants $\eta, K > 0$, $L, \mu, l \geqslant 0$ such that for each $x, y \in D_0$ the following conditions hold:*

- $\|A(x_0)^{-1}F(x_0)\| \leqslant \eta$,
- $\|A(x_0)^{-1}(F'(x) - F'(y))\| \leqslant K\|x - y\|$,
- $\|A(x_0)^{-1}(F'(x) - F'(x_0))\| \leqslant M\|x - x_0\| + \mu$,
- $\|A(x_0)^{-1}(A(x) - A(x_0))\| \leqslant L\|x - x_0\| + l$,
- $a = \mu + l < 1$,
- $h = \sigma\eta \leqslant \frac{1}{2}(1 - a)^2$, $\sigma = \max\{K, M + L\}$
*and*
- $\overline{S} = \overline{U}(x_0, t^*) \subseteq D_0$, *where*

$$t^* = \frac{1 - a - \sqrt{(1 - a)^2 - 2h}}{\sigma}.$$

*Then, sequence $\{x_n\}$ generated by Newton-like method*

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots, \quad (2.1)$$

*where $x_0$ is an initial pint is well defined, remains in $\overline{S}$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in \overline{S}$ of equation $F(x) = 0$. Moreover, the equation $F(x) = 0$ has a unique solution $x^*$ in $\overline{U}(x_0, t^*) \cap D_0$, if $h = \frac{1}{2}(1 - a)^2$, or in $\overline{U}(x_0, t^{**}) \cap D_0$, if $h < \frac{1}{2}(1 - a)^2$, where*

$$t^{**} = \frac{1 - a + \sqrt{(1 - a)^2 - 2h}}{\sigma}.$$

*Furthermore, for $\phi(t) = \frac{\sigma}{2}t^2 - (1 - a)t + \eta$ and $\psi(t) = 1 - Lt - l$ the following estimates hold for each $n = 0, 1, 2, \ldots$*

$$\|x_{n+1} - x_n\| \leqslant t_{n+1} - t_n$$

*and*

$$\|x_n - x^*\| \leqslant t_n - t^*,$$

*where scalar sequence $\{t_n\}$ is defined by*