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A new family of iterative methods widening areas of convergence ☆

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ABSTRACT

A new parametric class of third-order iterative methods for solving nonlinear equations and systems is presented. These schemes are showed to be more stable than Newton', Traub' or Ostrowski's procedures (in some specific cases), and it has been proved that the set of starting points that converge to the roots of different nonlinear functions is wider than the one of those respective methods. Moreover, the numerical efficiency has been checked through different numerical tests.

1. Introduction

Nonlinear equations or systems are usually involved in different problems in science and engineering. The analytical solution of these kind of problems is difficult and often iterative methods are used in order to estimate the solutions.

In the last decade, many multipoint Newton-type iterative schemes for solving nonlinear equations have appeared in the literature. A good survey about them can be found in [1]. Usually, these variants of Newton's method are designed to improve the original scheme in terms of order of convergence. However, they have some drawbacks: many of them are not applicable to nonlinear systems and high-order methods have quite reduced areas of convergence. We give emphasis, not as much to the order of convergence but to the region of good starting points of the methods and in the extension to nonlinear systems. So, our aim is to design a parametric family of iterative methods with wider areas of convergence than other known schemes and its generalization to multivariate cases.

In this work, we present a family of uniparametric two-point iterative procedure for solving the nonlinear equation $f(x) = 0$. It uses a damped Newton in the first step (acting as a predictor) and the corrector step is defined as a Newton-type scheme in which three functional evaluations are used. This corrector step is inspired in some modifications of Newton's method proposed by Ermakov and Kalitkin in [2]. Specifically, the authors presented a damped Newton scheme

$$x_{k+1} = x_k - \beta_k \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots, \tag{1}$$

where $\beta_k = \frac{|f(x_k)|^2}{|f(x_k)|^2 + \left|f\left(x_k - \frac{f(x_k)}{f'(x_k)}\right)\right|^2}$.

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Following this idea we present the parametric family of iterative schemes:

$$y_k = x_k - \alpha\frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,\ldots,$$
$$x_{k+1} = x_k - \frac{f(x_k)^2}{bf(x_k)^2 + cf(y_k)^2}\frac{f(x_k)}{f'(x_k)}, \tag{2}$$

where α, b and c are parameters. In the rest of the paper, we will denote this method by PM.

We prove that, under some conditions, the local order of convergence of the elements of the family is three. We can also extend this family for solving nonlinear systems $F(x) = 0$, holding the order of convergence.

We will compare the proposed schemes, in terms of the wideness of the regions of converging initial points, with the well-known Newton's method, whose iterative expression is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,\ldots. \tag{3}$$

It is known that this scheme converges quadratically in some neighborhood of the solution, under standard conditions. In comparisons, we will also use Traub's method (see [3]), which has order of convergence three (as our proposed schemes), whose iterative expression is

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\ x_{k+1} &= x_k - \frac{f(x_k)+f(y_k)}{f'(x_k)}, \quad k = 0,1,\ldots. \end{aligned} \tag{4}$$

In spite of the lower order of convergence of our proposed schemes, we will also compare them with Ostrowski's procedure [4], which iterative expression is

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\ x_{k+1} &= y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f'(x_k)}, \quad k = 0,1,\ldots, \end{aligned} \tag{5}$$

and whose order of convergence is four.

Let us note that expressions (1), (3) and (4) can be directly extended to nonlinear systems. On the other hand, recently, iterative expression (5) has been adapted for solving nonlinear systems [5].

In Section 2 we will analyze the local order of convergence of the designed schemes to find a simple root ξ of a nonlinear equation $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D. Moreover, the procedures will be extended in Section 2.2 for solving systems of nonlinear equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n, n > 1$. The dynamical behavior on different interesting equations will be studied in Section 3, by using distinct values of the parameter suitably chosen. This will allow us to select the most appropriate elements of the class in order to make the numerical test in Section 4, not only with nonlinear equations, but also with some nonlinear systems. Finally, we will state some conclusions and the references used.

2. Analysis of convergence

Firstly, we analyze the local order of convergence of the class of methods (2). Afterwards, we extend the mentioned family to nonlinear systems and we also study its convergence.

2.1. Study of the scalar case

In the next result, we show that there are values of parameters b and c, depending on α, such that the order of convergence of the resulting methods is three.

Theorem 1. *Let $\xi \in I$ be a simple zero of a sufficiently differentiable function $f : I \subset \mathbb{R} \to \mathbb{R}$ in an open interval I and x_0 an initial guess close enough to ξ. The method defined by (2) has third-order convergence if $b = \frac{1-\alpha+2\alpha^2}{2\alpha^2}$ and $c = \frac{1}{2\alpha^2(\alpha-1)}$, where $\alpha \notin \{0,1\}$. The error equation of the method is*

$$e_{k+1} = \frac{(-2+2\alpha+\alpha^2)c_2^2 + 2(-1+\alpha)^2 c_3}{2(-1+\alpha)}e_k^3 + O(e_k^4),$$

where $c_k = (1/k!)\frac{f^{(k)}(\xi)}{f'(\xi)}$, $k = 1,2,\ldots$ and $e_k = x_k - \xi$.