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On global dynamics in a periodic differential equation with deviating argument

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ABSTRACT

Several aspects of global dynamics and the existence of periodic solutions are studied for the scalar differential delay equation $x'(t) = a(t)f(x([t-K]))$, where $f(x)$ is a continuous negative feedback function, $x \cdot f(x) < 0, x \neq 0$, $0 < a(t)$ is continuous ω-periodic, $[\cdot]$ is the integer part function, and the integer $K \geqslant 0$ is the delay. The case of integer period ω allows for a reduction to finite-dimensional difference equations. The dynamics of the latter are studied in terms of corresponding discrete maps, including the partial case of interval maps ($K = 0$).

1. Introduction

Differential equations with deviating arguments (DEDAs) are an important part of modern theory and applications of nonlinear dynamical systems. Their theoretical fundamentals and multiple areas of applications were summarized in early works of the 60s and 70s, see e.g. monographs [1,11,27]. Most comprehensive theoretical achievements and numerous areas of applications were developed a few decades later and can be found in e.g. monographs [10,17]. The significance of new theoretical developments in the field and their wide applicability to modeling various real life phenomena have been overwhelmingly demonstrated in the past 20 years or so. The surge in research output during this time, both theoretical and numerical, was largely driven by applications, as can be seen from some recent review papers and monographs [12,21,22,31], where further related references can also be found. DEDAs are indispensable mathematical tools for modeling real life phenomena where after effects are intrinsic features of their functioning [13].

In this paper we study the global dynamics and, in particular, the existence and stability of periodic solutions, for a specific differential equation with piece-wise constant argument (see Eq. (1) below). This equation can be viewed as an approximation of a periodic differential delay equation with constant delay. Such an approximation allows for a reduction of its dynamics to that of an associated finite-dimensional map. Though the reduction is rather straightforward, the dynamics of the resulting map are highly non-trivial and not yet completely understood or studied. In a simpler partial case the dynamics of the DEDA are equivalent to that of a one-dimensional map. This case allows for a comprehensive study, the results of which are summarized in the present paper.

Consider the following differential equation with deviating argument

$$x'(t) = a(t)f(x([t-K])), \quad t \geqslant 0, \tag{1}$$

where the $[\cdot]$ is the integer value function, and the non-negative integer K is the delay. We shall assume throughout the paper that f is a continuous real-valued function satisfying the negative feedback condition

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$$x \cdot f(x) < 0 \quad \text{for all} \quad x \neq 0, \tag{2}$$

and is bounded from one side

$$f(x) \leqslant M \text{ or } f(x) \geqslant -M \text{ for all } x \in \mathbf{R} \text{ and some } M > 0. \tag{3}$$

The coefficient $a(t)$ is a continuous positive periodic function with integer period $\omega > 0$

$$a(t) > 0 \quad \text{and} \quad a(t+\omega) = a(t) \quad \text{for all} \quad t \in \mathbf{R}. \tag{4}$$

Eq. (1) can also be viewed as a differential equation with periodic delay.

Eq. (1) is closely related to the differential delay equation

$$x'(t) = a(t)f(x(t-\tau)), \tag{5}$$

with the same f and a, and where $\tau > 0$ is a constant delay. It can be viewed as a discrete version of Eq. (5). While the problem of global dynamics and the existence of periodic solutions for general Eq. (5) is quite difficult to approach, Eq. (1) appears to be somewhat simpler to study in this regard.

Eq. (1) falls within the class of differential equations with piecewise constant argument. Such equations have attracted a significant interest in recent years for their qualitative features and range of applications. As indicated by many authors, in general, numerical approximations of differential delay equations can give rise to DEDAs, see papers [5,6,15,24]. References to other applications can be found e.g. in [7,16,19,20]. Various aspects of their dynamics have been studied by many authors. Among those related to the present work we would like to mention papers [2,3,6,15,26,32].

When $a(t) = a_0 > 0$ is a constant Eq. (5) is equivalent to the well studied autonomous equation

$$x'(t) = G(x(t-1)). \tag{6}$$

It is well known that when G also satisfies the negative feedback condition (2), it is one-sided bounded in the sense of (3), and $G'(0) < -\pi/2$, then the differential delay Eq. (6) has a slowly oscillating periodic solution [10,17,18,25,28]. The proof of this fact is rather non-trivial; it constitutes a part of an established theory for the existence of periodic solutions of functional differential equations called ejective fixed point techniques [10,17,28].

It is a natural next step to look for the existence of non-trivial periodic solutions in similar but periodic functional differential equations of the form (5). As our initial approaches and analysis have indicated, the use of the standard techniques of the ejective fixed point theory does not appear immediately applicable to this case. New approaches and techniques seem to be necessary. Our first step in this direction is to study periodic solutions and other dynamical properties of the somewhat simpler differential delay Eq. (1). Another worthwhile avenue of possible approach to the study of periodic solutions of Eq. (5) is an extension and adaptation of techniques from [14] developed for a similar class of equations with almost periodic coefficients.

The assumption that the period ω is an integer is crucial for all principal considerations of the paper. It simplifies the dynamics of solutions of Eq. (1) significantly: they are essentially reduced to the dynamics of finite-dimensional discrete maps (which can be quite complex by themselves). At present we do not have a clear workable idea how to approach the case when the period ω is not commensurable with the delay in the equation (K for Eq. (1) and τ for Eq. (5)). Even in the simplest case of $K = 0$ and ω being irrational, the corresponding Eq. (1) seems to allow in some cases for the existence of quasi-periodic solutions.

2. Preliminaries

We shall be using throughout the paper the standard notions and definitions related to functional differential and difference equations, as well as to interval maps, most of which can be found in monographs [4,8–10,17,29,30].

For arbitrary initial function $\varphi \in \mathcal{C} := C([-K,0],\mathbf{R})$ the corresponding solution $x = x(t,\varphi)$ of Eq. (1) is easily found by successive integration for $t \geqslant 0$. One has

$$x(t) = \varphi(0) + f(\varphi(-K))\int_0^t a(s)ds \quad \text{for all} \quad t \in [0,1),$$

with

$$x(1) = \varphi(0) + a_1 \cdot f(\varphi(-K)), \quad \text{where} \quad a_1 := \int_0^1 a(s)ds.$$

Likewise

$$x(t) = x(1) + f(\varphi(-K+1))\int_1^t a(s)ds \quad \text{for all} \quad t \in [1,2),$$

with

$$x(2) = x(1) + a_2 f(\varphi(-K+1)), \quad \text{where} \quad a_2 := \int_1^2 a(s)ds,$$