Applied Mathematics and Computation 252 (2015) 473–483

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On continuous dependence of solutions of dynamic equations

Mieczysław Cichoń [a,*], Ahmet Yantir [b]

[a] Faculty of Mathematics and Computer Science, A. Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
[b] Department of Mathematics, Yaşar University, Izmir, Turkey

ARTICLE INFO

Keywords:
Time scale
Continuous convergence
Kuratowski limit of sets
Dynamic equation
Approximated solutions
Generalized exponential function

ABSTRACT

The main goal of the paper is to present a new approach to the problem of continuous dependence of solutions of differential or dynamic problems on their domains. This is of particular interests when we use dynamic (difference, in particular) equations as discretization of a given one. We cover a standard construction based of difference approximations for the continuous one, but we are not restricted only to this case. For a given differential equation we take a sequence of time scales and we study the convergence of time scales to the domain of the considered problem. We choose a kind of convergence of such approximated solutions to the exact solution. This is a step for creating numerical analysis on time scales and we propose to replace in such a situation the difference equations by dynamic ones. In the proposed approach we are not restricted to the case of classical numerical algorithms. Moreover, this allows us to find an exact solution for considered problems as a limit of a sequence of solutions for appropriate time scales instead of solving it analytically or calculating approximated solutions for the original problems.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction

When trying to find an exact or at least approximate solution for a given differential problem we can discretize it and then to find a solution as a limit of approximated ones. Usually this leads us to difference equations, i.e. discretized version of an original problem. Unfortunately, this approach has some restrictions or bad asymptotic properties. However, numerical methods are based on the latter method. We propose to put a more general starting point, by using an idea of dynamic equations instead of difference ones. We propose an algorithm based on the choice of a sequence of time scales $(\mathbb{T}_n)$ convergent to the domain of a considered problem. As long us we are able to solve the problem on each $\mathbb{T}_n$, we obtain an exact solution of the considered problem as a limit of the approximated ones. It means that by investigating the convergence of such solutions for problems defined on $\mathbb{T}_n$ we will treat the given limit as a solution of the original problem. We are interested in such a kind of dependence.

The main goal of the paper is to propose such a general method for solving differential (or dynamic) problems as a limit of approximated ones. To do this we need to study the continuous dependence results of solutions on their domains. We propose such a construction, thus comparing our approach with earlier ones. To achieve this goal we clarify the idea by investigating linear differential equations on a Cantor set (cf. [33], for instance). It is a good example of dynamic equations for which it is easier to find a sequence of dynamic equations approximating the original one rather than to solve it directly or to restrict our attention to difference problems. Our proposed approach should be treated as an extension of numerical

* Corresponding author.
E-mail addresses: mcichon@amu.edu.pl (M. Cichoń), ahmet.yantir@yasar.edu.tr (A. Yantir).

http://dx.doi.org/10.1016/j.amc.2014.12.047
0096-3003/© 2014 Elsevier Inc. All rights reserved.

analysis to the case of time scales. Moreover, it requires a definition of convergence of sequences of sets, i.e. time scales. Some continuous dependence of solutions on their domains are then necessary. We should present our procedure in such a way to cover the case of difference approximations of differential equations.

In last years, the theory of dynamic equations on time scales is still growing. Two main goals are considered in its foundation: the unification of separately investigated continuous and discrete problems and the discretization of continuous ones. Even though second goal is more interesting, surprisingly, most of papers are devoted to the first goal. Note that in some cases the "continuous" differential problem seems to be rather complicated, so the discretized version, i.e. the difference or the dynamic equation should be treated as an approximation for the original one.

There are two possible ways to do it. The first one is to construct a time scale approximating an original domain of the considered problem. In this case we need to have a nonempty intersection of domains for both problems and to compare the solutions on this common set – cf. [1], where the problem was announced and described. It is satisfactory solved on compact domains by extending all the problems to the case of real intervals (cf. also [19–21,6]).

The second approach is to take a sequence of time scales $(\mathbb{T}_n)$ convergent (in a precisely described sense) to the original domain of the problem $\mathbb{T}$ and then to investigate a sequence of solutions (x_n) of the problem on $\mathbb{T}_n$. In this case we need simultaneously approximate a point $t \in \mathbb{T}$ by some $t_n \in \mathbb{T}_n$ and a value of a solution $y(t)$ at this point, but it allows us to consider the general case of convergent sequences of time scales. A proper definition of convergence of sets is required. Moreover, we need to put the set of assumptions for each time scale separately. This approach will be described in our Section 2.2. Basic ideas in this direction can be found in [1,12,17,19,24,27,30,32], but (almost) all of that papers are devoted to study the case of compact time scales and the convergence (if any) is taken in the sense of the Hausdorff distance [1,19,21,27].

We are motivated by the study of some difference problems, so we focus on a specific kind of time scales (usually the Euler scheme with possibly variable time steps $z_{n+1} = z_n + h_n f(z_n)$. However, we are not restricted only to this case and we will investigate the general case. We stress on illustrative character of this paper by adding some examples. Although all the results are presented for so-called Δ-integrals, they can be easily adapted to the case of nabla-integrals and related concepts. Note that both mentioned concepts of integrals form approximations for differential equations and they are worthy of further research.

2. Time scales and dynamic equations

2.1. Basic notions

In this section, we briefly recall some basics about time scales and introduce some notations. The reader is directed to [7] for more detailed theory.

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. By $\mathbb{R}_+$ we denote the interval $[0, +\infty)$. Assume that $a < b$ are points in $\mathbb{T}$ and define the time scale interval $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum (q-difference) calculus, i.e. when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{Z}} = \{q^t : t \in \mathbb{Z}\}$, where $q > 1$. The Cantor set $\mathbb{K}$ is also a time scale and it will be discussed in Section 4.

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively.

We put $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). The jump operators σ and ρ allow the classification of points in time scale in the following way: t is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

The mapping $\mu : \mathbb{T} \to \mathbb{R}$ given by $\mu(t) = \sigma(t) - t$ will be called the graininess of t.

Let us recall that the dynamic equations on time scales are also useful when we try to unify continuous and discrete models (in biology, economics or in control theory, for instance). This is one more reason to propose an approach based on this idea (cf. [31]).

2.2. Convergence of time scales

We will prove an approximation property for a dynamic problem by a sequence of solutions for some "nicer" dynamic problems on variable domains. It will be preferable to approximate the domain for the problem rather than the function on the right-hand side of this problem. We are interested in describing both a convergence of time scales and a convergence of approximated solutions.

If we restrict our attention to the case of bounded time scales $\mathbb{T}_n \subset \mathbb{T}$, the uniform convergence on $\mathbb{T}_n$ (as in [1]) can be useful. If this not the case we can extend time scales $\mathbb{T}_n$ by considering $S_n = \mathbb{T}_n \cup \mathbb{T}$, but this means that the graininess function for the last time scale is different than for $\mathbb{T}_n$. In our opinion it is artificial. However, in some cases even this simple approach seems to be satisfactory and useful.

Download English Version:

https://daneshyari.com/en/article/4627161

Download Persian Version:

https://daneshyari.com/article/4627161

Daneshyari.com