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Convergence analysis of spectral collocation methods for a class of weakly singular Volterra integral equations ☆

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ABSTRACT

In this paper, we discuss the application of spectral Jacobi-collocation methods to a certain class of weakly singular Volterra integral equations. First, we use some function transformations and variable changes to transform the equation into a Volterra integral equation defined on the standard interval $[-1,1]$. Then the Jacobi–Gauss quadrature formula is used to approximate the integral operator. For the spectral Jacobi-collocation method, a rigorous error analysis in both the L^∞ and weighted L^2 norms is given under the assumption that both the kernel function and the source function are sufficiently smooth. Finally, some numerical examples are provided to illustrate the theoretical results.

1. Introduction

This paper is concerned with the weakly singular Volterra integral equation

$$y(t)=\int_0^t \frac{s^{\mu-1}}{t^\mu}K(t,s)y(s)ds+g(t),\quad 0\leqslant t\leqslant T, \tag{1.1}$$

where $\mu>0$, the kernel function $K(t,s)$ and the source function g are given smooth functions. This type of equations arise in certain heat conduction problems with time dependent boundary conditions (see, e.g., [15]). Note that the kernel is singular at $t=0$ and $s=0$ if $0<\mu<1$ and the kernel is singular at $t=0$ if $\mu\geqslant 1$.

From [10] we have the following analytical results of the solutions to (1.1) in the case $K(t,s)=1$.

Lemma 1.1. *(a) if $0<\mu\leqslant 1$ and $g\in C^1[0,T]$ (with $g(0)=0$ for $\mu=1$), the integral equation*

$$y(t)=\int_0^t \frac{s^{\mu-1}}{t^\mu}y(s)ds+g(t),\quad 0\leqslant t\leqslant T \tag{1.2}$$

has an infinite set of continuous solutions which are given by the formula

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$$y(t) = \begin{cases} c_0 t^{1-\mu} + g(t) + \frac{g(0)}{\mu-1} + t^{1-\mu}\int_0^t s^{\mu-2}(g(s)-g(0))ds, & \mu < 1, \\ c_0 t^{1-\mu} + g(t) + t^{1-\mu}\int_0^t s^{\mu-2}(g(s)-g(0))ds, & \mu = 1, \end{cases}$$

where c_0 is an arbitrary constant. The set of solutions contain only one particular solution which belongs to $C^1[0,T]$ (corresponding to $c_0 = 0$).

(b) if $\mu > 1$ and $g \in C^m[0,T]$ $(m \geqslant 0)$, the integral Eq. (1.2) possesses a unique solution $y \in C^m[0,T]$ given by

$$y(t) = g(t) + t^{1-\mu}\int_0^t s^{\mu-2} g(s)ds.$$

Several methods have been proposed in the literature to solve (1.2) for the case $\mu > 1$. An extrapolation algorithm based on the Euler method has been studied in [15]. In [7] high order product integration methods based on Newton–Cotes were proposed. Diogo et al. [9] developed a forth-order Hermite-type collocation method for problem (1.2). In [5] Diogo discussed the convergence properties of spline collocation and iterated methods for solving (1.2), and also the superconvergence properties of spline collocation based on the special choice of the collocation parameters were investigated in [8]. In recent years researchers have turned their attention to solving (1.2) with $0 < \mu \leqslant 1$. In [16], Lima and Diogo further developed the work carried out in [15] and obtained a convergence rate $O(h^\mu)$, where h stands for the maximum mesh size. Later Diogo et al. [6] obtained an improved convergence rate like $O(h/\epsilon)$, where ϵ is either a constant or a value depending on the mesh size h. In [17] the Euler method on graded meshes was used. To the best of our knowledge, very few works have been considered for (1.1) with general $K(x,t)$. In [11] the approximate solution of (1.2) was obtained by the optimal homotopy asymptotic method. In [1] Baratella derived a Nyström type interpolant of the solution based on Gauss–Radau nodes for solving (1.1).

Spectral methods are a widely used tool for solving several types of differential and integral equations [2,19]. It provides exceedingly accurate numerical results for smooth problems. In [20], Tang and Xu proposed a Legendre-collocation method and its error analysis is given for a Volterra integral equation of the second kind. In [12] the spectral method is used for a Volterra-type integro-differential equation. Chen and Tang [3,4] developed a spectral Jacobi-collocation method to solve the second kind Volterra integral equations (VIEs) with a weakly singular kernel $(t-s)^{-\mu}$ for $0 < \mu < 1$. Recently, Xie et al. [23] implemented spectral and pseudo-spectral Jacobi–Galerkin approaches for the second kind Volterra integral equation. In [21,22] Vainikko investigated polynomial collocation methods with Chebyshev knots for the cordial Volterra integral equation with a weakly singular kernel $t^{-1}\varphi(t^{-1}s)$. Recently, Jiang and Ma [13] studied spectral Chebyshev collocation methods for a class of Volterra integro differential equations with the same kernel as in (1.1).

We remark that Eq. (1.1) is different from the integral equation with a weakly singular kernel $(t-s)^{-\mu}$. For example, Eq. (1.1) has a smooth solution if $g(t)$ is smooth. In this work, we will consider this situation and develop high order numerical methods for (1.1). The spectral Jacobi-collocation method will be proposed for (1.1), and then a rigorous error analysis will be provided to theoretically justifies the spectral rate of convergence of the proposed method.

The remainder of the present paper is arranged as follows. In Section 2 the description of a spectral Jacobi-collocation method for the weakly singular Volterra integral equation (1.1) is given. Some lemmas useful for the convergence analysis are provided in Section 3. The convergence analysis in the L^∞-norm and the weighted L^2-norm will be performed in Section 4. Finally, we give some numerical experiments to confirm the theoretical results in Section 5.

2. Spectral Jacobi-collocation method

As defined in [2,19], the well-known Jacobi polynomials $J_n^{\alpha,\beta}(x),\ n = 0,1,\ldots$, with $\alpha,\ \beta > -1$ are the eigenfunctions of the singular Sturm–Liouville problem

$$-\frac{d}{dx}\left((1-x^2)\omega^{\alpha,\beta}\frac{d}{dx}J_n^{\alpha,\beta}(x)\right) = \lambda_n^{\alpha,\beta}\omega^{\alpha,\beta}J_n^{\alpha,\beta}(x).$$

The weight function $\omega^{\alpha,\beta}$ and eigenvalues $\lambda_n^{\alpha,\beta}$ are given as

$$\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta,$$
$$\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1).$$

Let the weighted space $L^2_{\omega^{\alpha,\beta}}(-1,1)$ be defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1) = \{u | u \text{ is measurable and } \|u\|_{\omega^{\alpha,\beta}} < \infty\},$$

equipped with the following inner product and norm

$$(u,v)_{\omega^{\alpha,\beta}} = \int_{-1}^1 u(x)v(x)\omega^{\alpha,\beta}(x)dx, \|u\|_{\omega^{\alpha,\beta}} = (u,u)^{\frac{1}{2}}_{\omega^{\alpha,\beta}}.$$

Then the set of Jacobi polynomials $J_n^{\alpha,\beta}(x),\ n = 0,1,\ldots$, forms a complete orthogonal basis of $L^2_{\omega^{\alpha,\beta}}(-1,1)$.