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Stability conditions for scalar delay differential equations with a non-delay term

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ABSTRACT

The problem considered in the paper is exponential stability of linear equations and global attractivity of nonlinear non-autonomous equations which include a non-delay term and one or more delayed terms. First, we demonstrate that introducing a non-delay term with a non-negative coefficient can destroy stability of the delay equation. Next, sufficient exponential stability conditions for linear equations with concentrated or distributed delays and global attractivity conditions for nonlinear equations are obtained. The nonlinear results are applied to the Mackey–Glass model of respiratory dynamics.

1. Introduction

Stability of the autonomous delay differential equation

$$\dot{x}(t) + bx(t-\tau) = 0 \tag{1.1}$$

(the sharp asymptotic stability condition for $\tau > 0$ is $0 < b\tau < \pi/2$) and of the equation with a non-delay term

$$\dot{x}(t) + ax(t) + bx(t-\tau) = 0 \tag{1.2}$$

was investigated in detail, and stability of (1.1) implies stability of (1.2) for any $a \geqslant 0$.

The equation

$$\dot{x}(t) + ax(t) + b(t)x(h(t)) = 0, \quad t \geqslant 0, \tag{1.3}$$

where $a > 0$ is a constant, b is a locally essentially bounded non-negative function, $h(t) \leqslant t$ is a delay function, is a generalization of (1.2) and also is a special case of the non-autonomous equation with two variable coefficients

$$\dot{x}(t) + a(t)x(t) + b(t)x(h(t)) = 0, \quad t \geqslant 0, \ a(t) \geqslant 0. \tag{1.4}$$

Let us note that, generally, asymptotic stability of the equation without the non-delay term

$$\dot{x}(t) + b(t)x(h(t)) = 0, \quad t \geqslant 0 \tag{1.5}$$

does not imply stability of (1.4).

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Example 1. Consider Eqs. (1.4) and (1.5) for $b(t) \equiv b > 0$ and $h(t) = [t]$, where $[t]$ is the maximal integer not exceeding t. The equation

$$\dot{x}(t) + bx([t]) = 0, \quad t \geqslant 0 \tag{1.6}$$

is asymptotically stable for any b satisfying $0 < b < 2$, since the solution on $[n, n+1]$ is $x(t) = x(n)[1 - b(t-n)]$ which is a linear function on any $[n, n+1]$. Thus $x(n) = (1-b)^n x(0)$ and $|x(n)| \leqslant \delta^n |x(0)|$, where $0 < \delta = |1-b| < 1$.

Let us choose $1.6 < b < 1.9$ and consider the equation

$$\dot{x}(t) + a(t)x(t) + bx([t]) = 0, \quad t \geqslant 0 \tag{1.7}$$

with a periodic piecewise constant non-negative function $a(t)$ with the period $T = 1$. If $a(t) \equiv \alpha$ on $[0, \varepsilon]$ for $0 < \varepsilon < 1$ then

$$x(t) = \left(\frac{b}{\alpha} + 1\right) x(0) e^{-\alpha t} - \frac{b}{\alpha} x(0), \quad t \in [0, \varepsilon].$$

Let us choose $\alpha = 3b$ and ε in such a way that $x(\varepsilon) = 0$, i.e. $\varepsilon = \frac{1}{3b} \ln 4$, and

$$a(t) = \begin{cases} 3b, & n \leqslant t \leqslant n + \varepsilon, \\ 0, & n + \varepsilon < t < n + 1, \end{cases} \tag{1.8}$$

where $n \geqslant 0$ is an integer. For $1.6 < b < 1.9$ we have $0.24 < \varepsilon < 0.29$, thus $|x(1)| = b|x(0)|(1-\varepsilon) > 1.136|x(0)|$. Further, $|x(n)| > 1.136^n |x(0)|$, which means that (1.7) is unstable, while (1.6) is asymptotically stable. Fig. 1, left, illustrates the solutions of (1.6) and (1.7) with $b = 1.8$, $x(0) = 1$, here $|x(n+1)| \approx 1.34|x(n)|$ for (1.7), so (1.7) is unstable while (1.6) is stable.

It is also possible to construct an example of asymptotically stable equation (1.6) with $a(t)$ satisfying $\inf_{t>1} a(t) > 0$ such that (1.7) is unstable. For example, consider

$$a(t) = \begin{cases} 3b, & n \leqslant t \leqslant n + \varepsilon, \\ 0.5, & n + \varepsilon < t < n + 1, \end{cases} \tag{1.9}$$

where $b = 1.8$, $x(0) = 1$. As previously, $x(t) = \frac{4}{3} x(n) e^{-\alpha(t-n)} - \frac{1}{3} x(n)$ on $[n, n+\varepsilon]$; the solution on $[n+\varepsilon, n+1]$ is $x(t) = 2bx(n)(e^{-0.5(t-n-\varepsilon)} - 1)$ and $|x(n+1)| \approx 1.12|x(n)|$ for (1.7). In this case $a(t) \geqslant 0.5$ for any t, and the solution is unstable and unbounded (see Fig. 1, right), though the divergence is slower than in the case when a is defined by (1.8).

For scalar differential equation (1.3), where $a > 0$ is a constant, b is a locally essentially bounded non-negative function, $h(t) \leqslant t$ is a delay function, the following result is a corollary of [1, Theorem 2.9].

Theorem 1. *Suppose* $0 \leqslant b(t) \leqslant b$, $0 \leqslant t - h(t) \leqslant h$ *and the inequality*

$$\frac{a}{b} e^{-ah} > \ln \frac{b^2 + ab}{b^2 + a^2} \tag{1.10}$$

holds. Then Eq. (1.3) *is exponentially stable.*

The aim of this paper is to extend Theorem 1 to other classes of equations, including (1.4), models with variable coefficients and several delays, as well as with distributed delays. In Section 3 we consider nonlinear delay differential equations and apply the results obtained to the Mackey–Glass model of respiratory dynamics in Section 4.

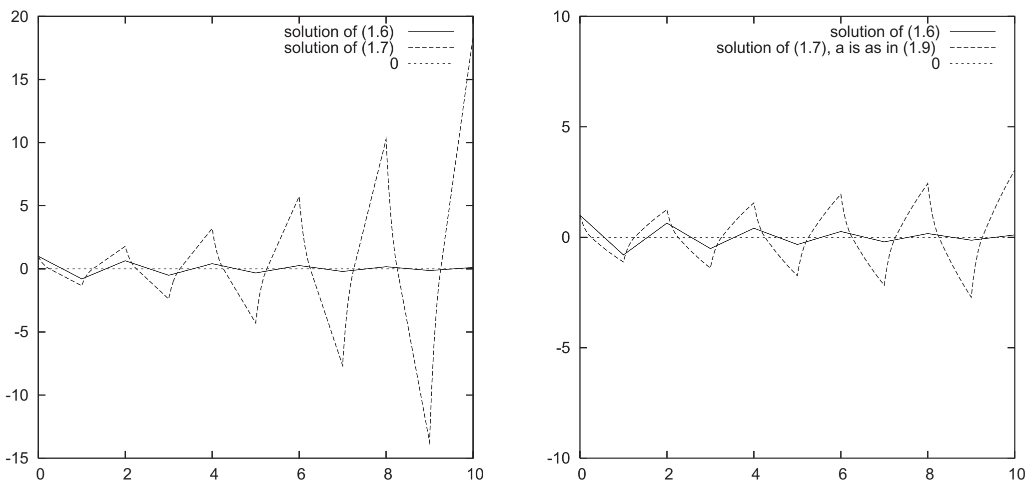

Fig. 1. Solutions of equations (1.6) and (1.7) with $b = 1.8$, $x(0) = 1$, $\varepsilon \approx 0.256721$ in the case when a is defined by (1.8) and can vanish (left) and a is described by (1.9) and satisfies $a(t) \geqslant 0.5$ (right). All the solutions are oscillatory, (1.6) is exponentially stable, while (1.7) is unstable in both cases.