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Real root classification of parametric spline functions ☆

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ABSTRACT

The real root classification of a given parametric spline function is a collection of possible cases of its real root distribution on every interval, together with the conditions of its coefficients must be satisfied for each case. This paper presents an algorithm to deal with the real root classification of a given parametric spline function. Two examples are provided to illustrate the proposed algorithm is flexible.

1. Introduction

The counting and classifying of the real/imaginary roots of a given polynomial have been the subject of many investigations. The classical Sturm Theorem and Tarski's Theorem are efficient methods for determining the numbers of real roots of polynomials with constant coefficients, but inconvenient for those with symbolic coefficients (see [1] for details). Fortunately, there are several different methods to determine the number of the distinct real roots of polynomials with symbolic coefficients. Yang et al. [2,3] established the complete discrimination system of a real parametric polynomial, which is sufficient for determining the numbers and multiplicities of the real/imaginary roots, namely, determining the complete root classification. In parallel, Gonzalez-Vega et al. [4] proposed the use of Sturm–Habicht sequences to solve the real roots of univariate polynomials. In 2009, Liang and Jeffrey [5] proposed automatic computation of the complete root classification for a real parametric polynomial. More importantly, the algorithm offered improved efficiency and a new test for non-realizable conditions.

Spline function (piecewise polynomial) is a natural generalization of polynomial. Thus, it is of theoretic and practical significance to study the counting and isolating the real roots of spline functions and its related problems. There exists several work on this issue [6–13]. For univariate case, Goodman [6] and de Boor [7] studied the relationship between the number of real roots of a univariate spline and the sequence of its B-spline coefficients, which provides new bounds on the number of real roots of the spline function. In 1989, Grandine [8] proposed a method for finding all real roots of a spline function based on the interval Newton method. In 2007, Morken and Reimers [9] presented an unconditionally quadratically convergent method for computing zeros of spline functions. In 2008, Wang and Wu [10] proposed an algorithm to isolate real roots of a given univariate spline based on the use of Descartes rule of signs with its B-spline coefficients and de Casteljau algorithm. For multivariate case, Lai et al. [11,12] gave the method to compute the supremum and its distribution of the distinct torsion-free real zeros of a given parametric piecewise polynomial system. In 2011, Wu and Zhang[13] presented an algo-

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rithm to isolate the real solutions of two piecewise algebraic curves based on the Krawczyk–Moore iterative algorithm. At the same time, Wu [14] presented an algorithm for computing the real intersection points of piecewise algebraic curves which is primarily based on the interval zeros of the univariate interval polynomial in Bernstein form. A question will arise: Can we count or classify of the real roots of a given spline function?

Given a set of spline knots $\Xi : -\infty \leqslant x_0 < x_1 < \cdots < x_N < x_{N+1} \leqslant \infty$. The univariate spline space $\mathbb{S}_n[x_0, x_1, \ldots, x_{N+1}]$ is defined as follows:

$$\mathbb{S}_n[x_0, x_1, \ldots, x_{N+1}] = \{S(x) \in C^{n-1} | S_i(x) \in \mathbb{P}_n[x], \quad i = 0, 1, \ldots, N\}, \tag{1}$$

where, $S_i(x)$ denotes the restriction of $S(x)$ over the interval $[x_i, x_{i+1}],\ i = 0, 1, \ldots, N$, and $\mathbb{P}_n[x]$ denotes the set of univariate polynomials with degree $\leqslant n$ in variable x. The function $S(x)$ is said to be of class C^{n-1} if and only if the derivatives $S'(x), S''(x), \ldots, S^{(n-1)}(x)$ exist and are continuous. Certainly, $\mathbb{S}_n[x_0, x_1, \ldots, x_{N+1}]$ is a linear space and its dimension is $n + N + 1$.

It is well-known that an arbitrary univariate spline $S(x) \in \mathbb{S}_n[x_0, x_1, \ldots, x_{N+1}]$ has the following unified representation [15]

$$S(x) = S_0(x) + \sum_{j=1}^{N} c_j (x - x_j)_+^n, \tag{2}$$

where truncation function means $x_+ = x$ for $x \geqslant 0$ and $x_+ = 0$ for $x < 0$, and $S_0(x)$ is a univariate polynomial with degree $\leqslant n$ on the initial interval $[x_0, x_1]$.

Firstly, we have the coarse bound of the number of real roots of a given univariate spline.

Theorem 1.1. *If $S(x) \in \mathbb{S}_n[x_0, x_1, \ldots, x_{N+1}]$, then the number of the roots of $S(x)$ is not greater than $n + N$.*

Proof. Obviously, the number of real roots of $S(x)$ on $[x_0, x_{N+1}]$ is equal to the summation of the number of real roots of $S_i(x)$ on $[x_i, x_{i+1}],\ i = 0, 1, \ldots, N$. Suppose the number of the real roots of $S(x)$ on $[x_0, x_{N+1}]$ is M, then $S'(x)$ has at least $M - 1$ real root. Inductively, $S^{(n-1)}(x)$ has at least $M - (n - 1)$ real roots. However, it is obvious that piecewise linear spline $S^{(n-1)}(x)$ has at most $N + 1$ real roots on $[x_0, x_N]$. Therefore, $M - (n - 1) \leqslant N + 1$, i.e., $M \leqslant n + N$.

It is pointed out that the "parametric" spline function means it contains symbolic coefficients and also allows to have some certain constant coefficients.

Secondly, we make the following two conventions for a given parametric spline.

- $S(x)$ is assumed to be "regular". It means that none of the spline knots is the real roots of $S(x)$, i.e.

$$S(x_i) \neq 0, \quad i = 0, 1, \ldots, N + 1. \tag{3}$$

- $S(x)$ is assumed to be non-degenerate. That's to say,

$$\deg(S_i(x)) = n, \quad i = 0, 1, \ldots, N, \tag{4}$$

where $S_i(x) = S_0(x) + \sum_{j=1}^{i} c_j (x - x_j)^n$. For example, if we give a parametric spline $S(x) = 2x^4 + bx^2 + c + 2(x - 1)_+^4 + d(x - 2)_+^4 \in \mathbb{S}_4[0, 1, 2, 3]$, then $S(x)$ has at most 6 real roots (counted with multiplicities) on (0,3) from Theorem 1.1 until now. In fact, we inevitably encounter another important problem that we want to know all the possible cases of the number of real roots of $S(x)$ and its distribution, as well as the conditions on these symbolic coefficients. □

Undoubting, one can deal with real root classification of piecewise polynomial on every interval individually. However, the spline function satisfies certain continuity on adjacent knots, we naturally ask a question: Can we tackle the real root classification of parametric splines on the whole? As the authors knowledge, there is no existing result on the real root classification of a given parametric spline till now. In this article, we mainly generalize the methods in papers [3,5] to solve the real root classification of a given univariate parametric spline. From now on, the parametric spline (2) is assumed to be regular and non-degenerate if not specified.

The rest of this paper is organized as follows. In Section 2, we recall several basic definitions and results on determining the number of real roots of a given parametric polynomial. In Section 3, we give the algorithm to tackle the real root classification of a given parametric spline, which is the main part of this paper. Finally, two illustrated examples are provided to show the algorithm is flexible in Section 4. Also, we conclude this paper in Section 5.

2. Preliminary

In this section, we shall mostly review the existing work with respect to the algorithm proposed by Yang et al. (see [3] for details and references therein). Let $f(x) \in \mathbb{R}[x]$ and write

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n.$$