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Generalized Golden Ratios defined by means

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ABSTRACT

Golden Ratio is defined by a proportion corresponding to the geometric mean. We introduce a generalized Golden Ratio as a fixed point of an operator defined by an arbitrary mean satisfying certain conditions. An algorithm for the evaluation of the generalized Golden Ratio is obtained using Banach's fixed point theorem. As applications we show that some problems in shape optimization correspond to some aesthetic standards introduced in the paper.

1. Introduction

The Golden Ratio appears probably for the first time in a problem formulated in Euclid's famous book "Elements", under the name of "extreme and mean ratio". There exist also theories stating that it was known even before Euclid's time. Since then, a great number of papers were written related to the subject. Its beautiful properties raised the interest of many authors, some of them becoming enthusiasts of the Golden Ratio (see for instance [11,15,16] and the references therein).

In the thirteenth century, Leonardo di Pisa (Fibonacci) studied the sequence that has today his name and is connected with the Golden Ratio [11].

Later, during the Italian Renaissance, the Golden Ratio was considered a very important aesthetic principle. In 1509 Fra Luca Pacioli, an Italian mathematician and painter who wanted to put some mathematical basis to painting, published the book "De Divina Proporzione". The illustrations of the book were made by Leonardo da Vinci.

The famous German astronomer Johannes Kepler compared it with a "precious jewel". The name Golden Section (Ratio) was probably used for the first time in nineteenth century by the German mathematician Martin Ohm.

In the nineteenth and twenty centuries numerous applications were emphasized in natural sciences (botany and zoology), music, encoding theory and of course, in architecture and painting (see [2,14]). Among the artists who used explicitly the Golden Ratio in their works we mention the painter Salvador Dali and the Swiss architect Le Corbusier, who proposed a scheme of hierarchical subdivisions based on it.

There exist also contrary points of view, several scientists consider that many of the cases where the Golden Ratio is claimed to appear in art or nature are exaggerated, since the measurements are not done according to some clear standards and is almost impossible to measure complex structures with enough accuracy (see for instance [5,12] and the references therein). Aesthetic properties and applications of the Golden Rectangle in architecture are also contested; the Golden Ratio could be rather used to architectural composition in the context of scaling hierarchy that organizes complexity ([1,13]).

Several generalizations of the Golden Ratio were defined, we mention here the papers [9,15].

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In what follows, by $[AB]$ we denote the closed line segment with endpoints A and B and by AB the length of $[AB]$. A point C divides a segment $[AB]$ in golden ratio if $C \in [AB], AC < CB$ and $\frac{AC}{CB} = \frac{CB}{AB}$.

This relation is equivalent to

$$CB = \sqrt{AC \cdot AB}, \tag{1}$$

i.e., CB is the geometric mean of AC and AB.

The number $\varphi = \frac{CB}{AC} = \frac{1+\sqrt{5}}{2}$ is called the golden ratio; φ is the positive solution of the equation

$$t^2 - t - 1 = 0. \tag{2}$$

Let us remark that (2) is the characteristic equation of the Fibonacci sequence $(F_n)_{n\geqslant 0}$ defined by the recurrence

$$F_{n+2} = F_{n+1} + F_n, \quad n \geqslant 0, \quad F_0 = 0, \quad F_1 = 1,$$

which is also connected with the golden ratio by the relation

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

For more details on the Fibonacci sequence and its relation with the Golden Ratio see for instance [4,6,7,10].

2. Golden Ratio defined by homogeneous means

According to the previous assertions a natural way for the extension of the Golden Ratio is to consider in (1), instead of the geometric mean, an arbitrary mean, satisfying certain conditions.

So, let $M(x,y)$ be a homogeneous mean, i.e., $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is a continuous function satisfying the relations

$$x < M(x,y) < y, \quad \text{for every } 0 < x < y, \tag{3}$$

$$M(\lambda x, \lambda y) = \lambda M(x,y), \quad \text{for every } \lambda, x, y \in (0,\infty). \tag{4}$$

We suppose that the mean M satisfies also the hypothesis:

(H) The equation

$$t = M(1, 1+t), \tag{5}$$

admits a unique solution in $(1,\infty)$.

Definition 1. Let $[AB]$ be a segment in the euclidian plane, $C \in [AB], CA = x, CB = y, x < y$. We say that C divides $[AB]$ in M-Golden Ratio (M-GR) if the following relation is satisfied

$$y = M(x, x+y). \tag{6}$$

According to the homogeneity of M the relation (6) is equivalent to

$$\frac{y}{x} = M\left(1, 1 + \frac{y}{x}\right), \tag{7}$$

which admits a unique solution $\frac{y}{x} := \varphi_M$, called the M-Golden Ratio.

Remark 1.

(1) If $M(x,y) = \sqrt{xy}, x,y \in (0,\infty)$ (the geometric mean) then the Eq. (5) becomes $t^2 - t - 1 = 0$ and $\varphi_M = \frac{1+\sqrt{5}}{2}$ is the classical Golden Ratio.
(2) If $M(x,y) = \frac{x+y}{2}, x,y \in (0,\infty)$ (the arithmetic mean) we get $\varphi_M = 2$.
(3) If $M(x,y) = \frac{2xy}{x+y}, x,y \in (0,\infty)$ (the harmonic mean) then $\varphi_M = \sqrt{2}$.

We give in what follows conditions under which the Eq. (5) admits a unique solution in the interval $(1,\infty)$ and an algorithm for the evaluation of φ_M, for an arbitrary mean.

Theorem 1. *Suppose that there exists $L \in [0,1)$ such that*

$$|M(1,x) - M(1,y)| \leqslant L|x - y|$$

for all $x,y \in [2,\infty)$. Then the Eq. (5) admits a unique solution $\varphi_M \in (1,\infty)$. The sequence $(t_n)_{n\geqslant 0}$ given by the relation

$$t_{n+1} = M(1, 1+t_n), \quad n \geqslant 0, \quad t_0 \geqslant 1, \tag{8}$$