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Applications of the Numerical Inversion of the Laplace transform to unsteady problems of the third grade fluid

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ABSTRACT

In this article, we have effectively used the Numerical Inversion of Laplace transform to study some time dependent problems of the third grade fluid. To do so, we have considered three different types of unsteady flows of the third grade fluid, namely

(a) Unsteady flow over a flat rigid plate with porous medium.
(b) Unsteady MHD flow in a porous medium.
(c) Unsteady MHD flow in a non-porous space with Hall currents.

The solution to the governing equation in each case is obtained by using the standard Laplace transform. However, to transform the obtained solutions from Laplace space back to the original space, we have used the Numerical Inversion of the Laplace transform. Graphical results for each case have been presented to show the effects of different parameters involved and to show how the fluid flow evolves with time.

1. Introduction

Laplace transform is a very useful tool for solving the differential equations. However, to analytically compute the inverse Laplace transform of the solutions obtained by the use of the Laplace transform is a very important but complicated step. To overcome this issue, several algorithms for Numerical Inversion of Laplace transform have been proposed in literature [1–4]. Here we implement the idea of Numerical Inversion of Laplace transform presented by Weeks [1] and the one by Juraj and Lubomir [2]. The inverse Laplace transform $G(s)$ of a function $g(t)$ is given by the following contour integral

$$g(t) = \frac{1}{2\pi i} \lim_{\omega\to\infty} \int_{p-i\omega}^{p+i\omega} G(s)e^{st}ds. \tag{1}$$

Here the Laplace variable $s = p + i\omega$ and p is greater than the real part of any singularity in the transformed function $G(s)$. Weeks method [1] for Numerical Inversion of Laplace transform is based on the use of the Laguerre functions. The method of Juraj and Lumboir [2] is based on the use of infinite series for $\cosh(z)$ and $\sinh(z)$ along with the application of residue theorems to compute the contour integral given in (1).

Non-Newtonian fluids have been a famous topic of research because of their diverse use in many industrial processes. Various complex fluids such as Oils, polymer melts, different type of drilling muds and clay coatings and many emulsions are included in the category of non-Newtonian fluids. Flows of non-Newtonian fluids have been studies by many [5–21]. One of the very important models suggested for non-Newtonian fluids is called the third-grade fluid model [22,23]. However,

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studying unsteady problems of the third grade fluids have always been a difficult and challenging task. Here, we solve some unsteady problems of the third grade fluid by using the Laplace transform and then implement the algorithms of Numerical Inversion of Laplace transform to obtain the solutions in the original space and to find the effects of different fluid parameters. It also helps us to understand the evolution of the flow with time. This work is a continuation of our earlier work [5] where we used the Numerical Inversion of the Laplace transform to study a thin film flow of a second grade fluid through a porous medium. Now we implement the same numerical algorithms [1–4] to make a study of different cases involving a third grade fluids.

2. Basic equations

The momentum balance equation in the presence of a body force B and the equation of conservation of mass are respectively given as

$$\rho\frac{D\vec{v}}{Dt} = div(\vec{\tau}) + (\mathbf{J}\times\mathbf{B}), \tag{2}$$

$$\nabla\cdot\vec{v} = 0, \tag{3}$$

with ρ being the density, $\vec{v}$ the velocity, $\vec{\tau}$ the stress tensor and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$ represents the material derivative. The symbol $\mathbf{J}$ is the current density and $\mathbf{B} = (\mathbf{B}_0 + b)$ is the total magnetic field (normal to the velocity), $\mathbf{B}_0$ is the applied magnetic field and b is the induced magnetic field. The stress tensor $\vec{\tau}$ in Eq. (2) for a third grade fluid has the form

$$\vec{\tau} = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2(A_1A_2 + A_2A_1) + \beta_3\left(\mathrm{tr}A_1^2\right)A_1, \tag{4}$$

where I is the identity tensor, p is the pressure, μ is the dynamic viscosity, α_1 is the elastic coefficient and α_2 is the transverse viscosity coefficient, β_1, β_2 and β_3 are material constants for the third grade fluid and A_1, A_2 and A_3 represent the Rivlin–Ericksen tensors. The tensors A_1 is defined by the following expression

$$A_1 = (\nabla\vec{v}) + (\nabla\vec{v})^T, \tag{5}$$

where A_2 and A_3 are defined as

$$A_{n+1} = \frac{dA_n}{dt} + A_n(\nabla\vec{v}) + (\nabla\vec{v})^T A_n, \quad n = 2, 3, \tag{6}$$

with

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3}, \quad \beta_3 \geqslant 0. \tag{7}$$

The part $\mathbf{J}\times\mathbf{B}$ in Eq. (2) will only be relevant for MHD flows. The Maxwell equations and the generalized Ohm's law are given as

$$\mathrm{div}\ \mathbf{B} = 0, \quad \mathrm{Curl}\ \mathbf{B} = \mu_m\mathbf{J}, \quad \mathrm{Curl}\ \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{8}$$

$$\mathbf{J} + \frac{w_e\tau_e}{\mathbf{B}_0}\mathbf{J}\times\mathbf{B} = \sigma(\mathbf{E} + \mathbf{J}\times\mathbf{B}). \tag{9}$$

Here σ and μ_m are constants through the flow field. w_e and τ_e are the cyclotron frequency and collision time of the electrons, respectively. We will assume that $w_e\tau_e \approx O(1)$ and $w_i\tau_i \ll 1$ where the subscript i corresponds to ions. Also, following [24], for a unidirectional third grade fluid through porous medium, the relationship between the pressure and the velocity is given by

$$\frac{\partial p}{\partial x} = -\frac{\phi}{\kappa}\left[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]u, \tag{10}$$

where κ is the permeability and ϕ the porosity of the porous space.

3. Unsteady flow of a third grade fluid over a flat rigid plate through a porous medium

We consider the flow of a third grade fluid through a porous medium [24]. We consider $\mathbf{v} = [u(y,t), 0, 0]$ to be the velocity profile for such a flow. Under these considerations, the governing equations for this case become

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2\partial t} + 6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} - \frac{\phi}{\kappa}\left[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]u. \tag{11}$$