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Homoclinic solutions for a nonperiodic fourth order differential equations without coercive conditions ☆

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ABSTRACT

In this paper we investigate the existence of homoclinic solutions for the following fourth order nonautonomous differential equations

$$u^{(4)} + wu'' + a(x)u = f(x,u), \qquad \text{(FDE)}$$

where w is a constant, $a \in C(\mathbb{R},\mathbb{R})$ and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. The novelty of this paper is that, when (FDE) is nonperiodic, i.e., a and f are nonperiodic in x and assuming that a does not fulfil the coercive conditions and f satisfies some more general (AR) condition, we establish one new criterion to guarantee that (FDE) has at least one nontrivial homoclinic solution via using the Mountain Pass Theorem. Recent results in the literature are generalized and significantly improved.

1. Introduction

In the present paper we deal with the existence of homoclinic solutions for the following nonperiodic fourth order nonautonomous differential equations

$$u^{(4)} + wu'' + a(x)u = f(x,u), \tag{FDE}$$

where w is a constant, $a \in C(\mathbb{R},\mathbb{R})$ and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. In (FDE), let $f(x,u)$ be of the form

$$f(x,u) = b(x)u^2 + c(x)u^3,$$

then (FDE) reduces to the following equation

$$u^{(4)} + wu'' + a(x)u - b(x)u^2 - c(x)u^3 = 0, \tag{1.1}$$

which has been put forward as mathematical model for the study of patten formation in physics and mechanics. For example, the well-known Extended Fisher–Kolmogorov (EFK) equation proposed by Coullet et al. [6] in study of phase transitions, and also by Dee and van Saarlos [7], as well as the Swift–Hohenberg (SH) equation [15] which is a general model for pattern-

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forming process derived to describe random thermal fluctuations in the Boussinesq equation and in the propagation of lasers [8]. With appropriate changes of variables, stationary solutions of these equations lead to the following fourth order equation

$$u^{(4)} + wu'' - u + u^3 = 0, \tag{1.2}$$

where $w > 0$ corresponds to (EFK) equation and $w < 0$ to (SH) equation. In addition, in the description of water waves driven by gravity and capillarity [4], the following differential equation can be reduced by means of an argument based on the center manifold theorem

$$u^{(4)} + wu'' - u + u^2 = 0, \tag{1.3}$$

where $w < 0$ is a constant. Meanwhile, in study of weak interactions of dispersive waves, Bretherton [2] gave the following partial differential equation

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial t^4} + v - v^3 = 0.$$

To obtain traveling wave solutions $v(t,x) = u(x - ct)$ with $c > 0$, one can deduce that

$$u^{(4)} + c^2 u'' - u + u^3 = 0. \tag{1.4}$$

Besides, pulse propagation through optical fibers involving fourth order dispersion leads to a generalized nonlinear Schrödinger equation [5]

$$i\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial t^2} - \frac{\partial^4 v}{\partial t^4} + |v|^2 v = 0.$$

Assuming that harmonic spatial dependence $v(t,x)$ be of the form $v(t,x) = u(t)e^{ikx}$ $(k \in \mathbb{N})$, then one obtains

$$u^{(4)} - u'' + ku - u^3 = 0. \tag{1.5}$$

In what follows, for the reader's convenience we give some brief description for the study of homoclinic solutions of the fourth order differential equations. For the autonomous case, if $a(x) = b(x) = 1,\ c(x) = 0$ and $w \le 2$, Amick and Toland [1] proved the existence of homoclinic solutions of Eq. (1.1). Later, their result was extended by Buffoni in [3]. If $a(x) = c(x) = 1,\ b(x) = 0$, Peletier and Tory [12] extensively studied the periodic, homoclinic and heteroclinic solutions of Eq. (1.1).

Compared to the autonomous case, the nonautonomous case seems to be more difficult, because of the lack of the translation invariance and the existence of a first integral. Tersian and Chaparova [15] showed that Eq. (1.1) possesses one nontrivial homoclinic solution by using the Mountain Pass Theorem when $a(x),\ b(x)$ and $c(x)$ are continuous periodic functions and satisfy some other assumptions. Li [10] extended the results to some general nonlinear term, i.e., (FDE), assuming that $a(x)$ and $f(x,u)$ are periodic in x, and $f(x,u)$ satisfies the following Ambrosetti–Rabinowitz condition ((AR) condition):

(AR) there is a constant $\vartheta > 2$ such that

$$0 < \vartheta F(x,u) \leqslant f(x,u)u, \quad \forall (x,u) \in \mathbb{R} \times \mathbb{R},$$

where $F(x,u) = \int_0^u f(x,t)dt$.

Moreover, Li [11] dealt with the nonperiodic case of Eq. (1.1) and obtained the existence of nontrivial homoclinic solutions via using a compactness lemma and a Mountain Pass Theorem. Sun and Wu [14] considered the following nonperiodic fourth order differential equations with a perturbation

$$u^{(4)} + wu'' + a(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \quad x \in \mathbb{R},$$

where w is a constant, $\lambda > 0$ is a parameter, $a \in C(\mathbb{R},\mathbb{R}),\ f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R}),\ 1 \leqslant p < 2$ and $h \in L^{\frac{2}{2-p}}(\mathbb{R})$. More recently, Li, Sun, etc. [9] studied the existence of infinitely homoclinic solutions for nonperiodic (FDE) when the nonlinear term $f(x,u)$ satisfies the superlinear condition, but not assuming the well-known (AR) condition, see Theorem 1.1 in [9]. However we must point out that, for the case that (FDE) is nonperiodic, to obtain the existence of homoclinic solutions, the following coercive condition on a is often needed:

(a) $a : \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exists some constant $a_1 > 0$ such that

$$0 < a_1 \leqslant a(x) \to +\infty \quad \text{as} \quad |x| \to +\infty, \tag{1.6}$$

which is used to establish the corresponding compact embedding lemmas on suitable functional spaces, see Lemma 2 in [11], Lemma 2.2 in [14] and Lemma 2.2 in [9]. Clearly, if a is bounded, then it is not covered by (a). Consequently, a natural question is that how to obtain the existence of homoclinic solutions of (FDE) for the case that a is nonperiodic and bounded.

In the present paper we focus our attention on the case that (FDE) is nonperiodic and a is bounded in the following sense

($\mathcal{A}$) $a \in C(\mathbb{R},\mathbb{R})$ is continuous and there exist two constants $0 < \tau_1 < \tau_2 < +\infty$ such that