Applied Mathematics and Computation 250 (2015) 479–491

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Geometry structures of the generalized inverses of block two-by-two matrices ☆

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ARTICLE INFO

*Keywords:*
Block matrix
Inverse
Moore–Penrose inverse

ABSTRACT

In this paper, we study the invertibility and the Moore–Penrose invertibility of $2 \times 2$ block matrix with Schur complement having closed range. The necessary and sufficient conditions for the existence as well as the expressions for the inverses and the Moore–Penrose inverses are obtained.



## 1. Introduction and preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be separable, infinite dimensional, complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $B(H,K)$ and by $B(H)$ when $\mathcal{H} = \mathcal{K}$. For $A \in B(H,K)$, let $A^*, \mathcal{R}(A)$ and $\mathcal{N}(A)$ be the adjoint, the range and the null space of $A$, respectively. If $\mathcal{M}$ is a subspace of a Hilbert space, the dimension of $\mathcal{M}$ is denoted by dim $\mathcal{M}$. For $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, if there exists an operator $T^+ \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying the following four operator equations

$$TT^+T = T, \quad T^+TT^+ = T^+, \quad TT^+ = (TT^+)^*, \quad T^+T = (T^+T)^*, \tag{1}$$

then $T^+$ is called the Moore–Penrose inverse (for short MP-inverse) of $T$. It is well known that $T$ has the MP inverse if and only if $\mathcal{R}(T)$ is closed and the MP inverse of $T$ is unique [2,4,19,26]. We use $T^{\{1\}}$ to denote an arbitrary solution to the operator equation $TXT = T$. In general, $T^{\{1\}}$ is not unique and $(STK)^{\{1\}} = K^{-1}T^{\{1\}}S^{-1}$ for arbitrary invertible operators $S$ and $K$. The symbols $E_A = I - AA^+$ and $F_A = I - A^+A$ stand for the orthogonal projectors onto $\mathcal{R}(A)^\perp$ and onto $R(A^*)^\perp$, respectively. The identity onto a closed subspace $\mathcal{M}$ is denoted by $I_{\mathcal{M}}$ or $I$ if there is no confusion. An operator $A$ is said to be positive if $(Ax,x) \geqslant 0$ for all $x \in \mathcal{H}$. If $A$ is positive, the positive square root of $A$ is denoted by $A^{\frac{1}{2}}$ [5,22]. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be an orthogonal projector if $P^2 = P = P^*$. The orthogonal projector onto a closed subspace $U$ is denoted by $P_U$. Let $\overline{\mathcal{M}}$ denote the closure of $\mathcal{M}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $P^2 = P$.

The MP-inverse has been proved helpful in systems theory, difference equations, differential equations and iterative procedures. It would be useful if these results could be extended to infinite dimensional situations. Applications could then be made to denumerable systems theory, abstract Cauchy problems, infinite systems of linear differential equations, and possibly partial differential equations and other interesting subjects [3,15].

Let $M$ be the block operator matrix from $\mathcal{H}_1 \oplus \mathcal{H}_2$ into $\mathcal{K}_1 \oplus \mathcal{K}_2$ over the field of complex numbers

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{pmatrix}. \tag{2}$$

☆ Supported by the National Natural Science Foundation of China under Grant 11171222.
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http://dx.doi.org/10.1016/j.amc.2014.11.008
0096-3003/

If $A$ is MP-invertible, then $S_A = D - CA^+B$ stands for the generalized Schur complement of $A$ in $M$. It is obvious that, under suitable partitioning, any operator can be cast in the form (2). One of the fundamental and important problems is how to examine the invertibility or generalized invertibility. In this paper, we discuss the sufficient and necessary conditions that guarantee the invertibility and MP-invertibility of a $2 \times 2$ block operator valued matrix $M$ with one of subblocks $A, B, C$ and $D$ having the closed range. The general formulae for the inverses and MP-inverses of $M$ are derived. Applying these results, we can obtain the inverse and MP-inverses of $2 \times 2$ block operator valued matrices with specified properties.

As we know, if $\mathcal{R}(A)$ is closed, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^+ & I \end{pmatrix} \begin{pmatrix} A & E_A B \\ CF_A & S_A \end{pmatrix} \begin{pmatrix} I & A^+B \\ 0 & I \end{pmatrix}.$$

Many authors have considered this problem and presented formulae for the inverses or MP-inverses under specific conditions [1,20,23,25]. We list some cases. When $A$ is invertible, by block Gaussian elimination of $M$ [18, Exercise 2.6.15],

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & S_A \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix},$$

it is well-known that $M$ is invertible if and only if $S_A$ is invertible. The inverse of $M$ is

$$M^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & S_A^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}. \tag{3}$$

The expression (3) is called the Banachiewicz–Schur form of $M$ and can be found in standard textbooks on linear algebra [12]. When

$$M_0 = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

is positive semidefinite, there exists a contraction operator $T$ (i.e., $\|T\| \leqslant 1$) such that $B = A^{\frac{1}{2}}TD^{\frac{1}{2}}$ [10] and

$$M_0 = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & D^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & 0 \\ T^* & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I - T^*T \end{pmatrix} \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & D^{\frac{1}{2}} \end{pmatrix}.$$

So $M_0$ is invertible if and only if $A, D$ are invertible and there exists $T$ with $\|T\| < 1$ such that $B = A^{\frac{1}{2}}TD^{\frac{1}{2}}$. The inverse of $M_0$ is

$$M_0^{-1} = \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & D^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & -T \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - T^*T)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -T^* & I \end{pmatrix} \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & D^{-\frac{1}{2}} \end{pmatrix}.$$

When $A, B, C, D$ are pairwise commutative operators, $M$ is invertible if and only if $AD - BC$ is invertible [13, Problem 70]. When $C, D$ are commutative and $D$ is invertible, then $M$ is invertible if and only if $AD - BC$ is invertible.

The paper is organized as follows. In Section 2, we list several properties of range relations of operators and some formulae for the MP-inverses of linear bounded operators or $2 \times 2$ block operator matrices. In Section 3, we apply these results to get the inverses of $2 \times 2$ block matrices with one of blocks having closed range. In Section 4, we investigate the necessary and sufficient conditions which insure that a $2 \times 2$ block matrix is MP-invertible and obtain the detailed MP-inverse formulae by certain structures.

## 2. Some lemmas

In this section we shall begin with some lemmas. We need the following well-known criteria about ranges. The following item (i) is from [11, Theorem 2.2].

**Lemma 2.1** ([9,16], [11, Theorem 2.2]). *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

(i) $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{\frac{1}{2}})$.
(ii) *$\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(AA^*)$ if and only if $\mathcal{R}(A^*)$ is closed.*
(iii) *If $S$ and $T$ are invertible, then $\mathcal{R}(SAT)$ is closed if and only if $\mathcal{R}(A)$ is closed.*
(iv) *If $A \geqslant 0$ is a positive operator, then*
  (a) $\overline{\mathcal{R}(A^{\frac{1}{2}})} = \overline{\mathcal{R}(A)}$, $\mathcal{R}(A) \subseteq \mathcal{R}(A^{\frac{1}{2}})$.
  (b) *$\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$.*
  (c) *$\mathcal{R}(A) = \mathcal{H}$ if and only if $A$ is invertible.*
(v) *If $A, B$ are MP-invertible, $AB^* = 0$ and $A^*B = 0$, then $A + B$ is MP-invertible and $(A + B)^+ = A^+ + B^+$.*

The next result was proved in [17] in the setting of rings. It still holds for bounded operators.